BASIC MATHEMATICS for Technical Courses

Second Edition

CHECKED

by

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PREFACE TO THE SECOND EDITION

THIS second edition has been prepared to meet the needs and incorporate the suggestions voiced in requests of users of the original edition.

The solution of simultaneous equations by determinant notation has been extended to take care of problems with more than three equations because of the frequency with which such problems appear in the engineering field. Synthetic division and synthetic substitution in equation solution have been included as additional helpful short cuts. Material on polar vectors has been added to that on rectangular vectors because of the frequent use of polar vectors in electrical circuits. Many review problems have been added at the end of each chapter.

CLARENCE E. TUITES

PREFACE TO THE FIRST EDITION

THE material used in this book was developed at the Rochester Institute of Technology and was used for several years in mimeographed form by the author in his classes at that institute. It was selected by the publisher as one of a series of textbooks designed for technical institutes and the junior college field. It should be useful also in industrial and extension schools and to those who must depend upon self-study for the continuation of their education.

The book provides the mathematical training that is pertinent and essential in the study of technical subjects. It was written from the viewpoint that mathematics in the technical field should be a means to an end rather than an end in itself. Since the theory of mathematics is combined closely with its use, applications in the solution of practical problems have been given prominence. Concise analysis rather than mere substitution in a formula has been emphasized as the proper approach to the problem solving, and therefore students are encouraged to use diagrams, whenever possible, as an aid in the thinking process.

Representative problems from the various technical fields have been selected so that a student with a particular interest in one of these fields may be given an assignment of practice problems illustrative of the use of mathematics in his chosen field. A number of problems are presented in the form of mechanical drawing, since such problems frequently occur in technical work. Assignments may be made for either long or short courses. A larger number of problems are included than the average instructor will require unless a longer course is desired.

The use of the operator j is introduced in place of the imaginary i, which is customarily used in mathematics, to conform to engineering practice, where the symbol i is used to represent other quantities. Vectors and components are treated at length because they are used extensively in the various engineering fields. Likewise, the solution of simultaneous equations by determinant notation is emphasized because of the value of this method in some types of engineering problems.

Since many engineering problems are solved with the slide rule, the student should acquire facility in its use at the beginning of his mathematics course rather than later. For this reason the study of how to use a slide rule is presented in the first chapter. The explanation of the theory of the slide rule is not given until the work in logarithms has been studied.

In the opinion of the author, a knowledge of the sine law and the cosine law combined with the use of the slide rule and a table of squares and roots is adequate for the solution of the oblique triangle. Therefore, the law of tangents and the half-angle formulas are not discussed until Chapter 6 of Part II, and they may be omitted at the discretion of the instructor.

CLARENCE E. TUITES

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Mr. Dean Frederick and Mr. Kenneth French, former instructors in the Electrical Department at the Rochester Institute of Technology, and Mr. Hollis Todd, instructor in the Photographic Technology Department at the Rochester Institute of Technology, have contributed many helpful suggestions.

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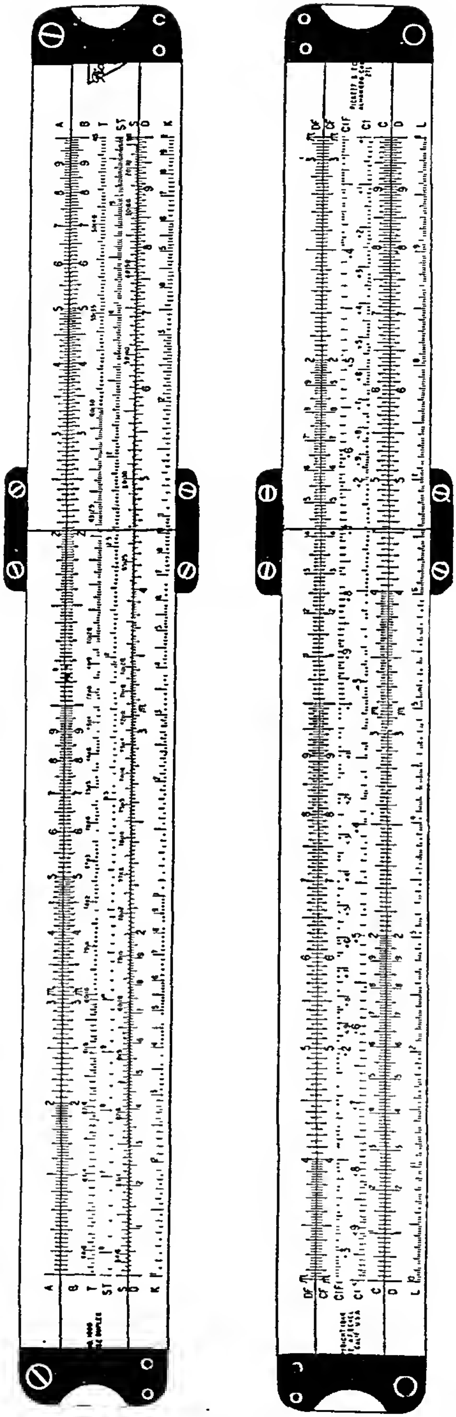

Pickett and Eckel Slide Rule No. 1000.

Chapter 1

THE SLIDE RULE

1. Use of the slide rule. The slide rule is an instrument designed to save time and labor in nearly all mathematical calculations encountered in everyday experiences. It is particularly valuable to the technical man because his work is based on mathematics. Briefly, it is an instrument to aid in performing the various fundamental operations of multiplication, division, squaring, cubing, and extracting square roots and cube roots in much less time and with much less effort than it can be done by ordinary methods.

There are many types of slide rules, but there are certain characteristics common to all of them that must be studied because they form the basis on which the operation of the rule is founded. Further, it is essential to learn to read the rule correctly before attempting to solve problems.

2. C and D scales of the slide rule. There are four scales, A, B, C, and D, common to all slide rules, but we shall consider only the C and D scales at the moment, since they are used for the more simple operations of multiplication and division.

It will be noted that the C and D scales are exactly alike; that is, each scale is numbered from 1 (called the *left index*) on the extreme left through the numbers (in large type) 2, 3, 4, 5, 6, 7, 8, and 9, successively, to 1 (called the *right index*) on the extreme right, with the spaces between the figures decreasing steadily toward the right. The figure 1 on the left may stand for any number in which 1 is the only figure, other than zero. Thus, it may stand for 1, 10, 100, 1,000; or for 0.1, 0.01, 0.001, and so on. If this 1 on the left stands for unity, then the 1 on the right will be 10; but if the 1 on the left is 10, then the 1 on the right is 100, and if the 1 on the left is 100, then the 1 on the right is 1,000, and so on. Fig. 1–1 illustrates these *main divisions* on the C and D scales.

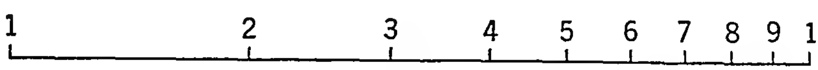

Fig. 1–1.

It is evident, then, that the *main division* numbering will depend upon what is chosen for the left index. Thus, if this index is unity, then the numbering reads 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. But if this left index is 10, then the numbering is 10, 20, 30, 40, 50, 60, 70, 80, 90, and 100. Again, if the index is 0.1, then the numbering will be 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0.

Now consider the space between the left index and the large figure 2. It will be noted that this space is divided off into 10 smaller spaces that decrease steadily toward the right and that the division points between these spaces are numbered from 1 to 9, inclusive, with smaller type. These spaces are called *secondary divisions* and are illustrated in Fig. 1–2.

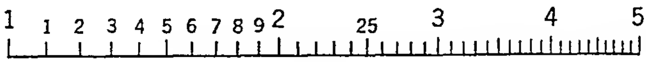

Fig. 1–2.

It is plain that the significance of the numbers for these secondary divisions will depend also upon what is chosen for the left index. Thus, if the left index is 1 or unity, then this numbering will be read 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, and 2. But if the left index is 10, then these numbers will be read 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20. The spaces between 2 and 3, 3 and 4, 4 and 5, 5 and 6, 6 and 7, 7 and 8, 8 and 9, 9 and 1, are also divided off into the same secondary divisions as the space between 1 and 2, although the smaller-type figures are not included. Therefore, we may now say that the rule is divided into 10 main divisions and each main division is divided into 10 secondary divisions.

Again, looking at the secondary divisions in the main division between 1 and 2, it will be seen that each secondary division is divided into 10 parts or *subdivisions*, thus giving a value of unity or 1 to each of these subdivisions. However, as the main divisions progress toward the right of the rule, they become increasingly smaller, and it is not possible to divide all their secondary divisions into 10 parts as was done with those in the main division between 1 and 2. Therefore, it will be noted that the secondary divisions between 2 and 3 and between 3 and 4 are divided into 5 subdivisions each, thus giving a double value or a value of 2 to each of these subdivisions; and for the remainder of the scale the secondary divisions are divided into 2 subdivisions each, thus giving a value of 5 to each subdivision.

Now to locate a 3-digit number on the C or D scale, the following procedure is necessary:

First. Read the first significant figure, which is the first numeral that is not zero. If this figure is 1, then the number lies between the main

Fig. 1–3.

divisions 1 and 2; if the figure is 2, then the number lies between the main divisions 2 and 3; and so on. Thus, on the skeleton scale shown in Fig. 1–3 the number 157 will be found between the main divisions 1 and 2, since its first significant figure is 1.

Second. Read the second figure, which will locate the number on the secondary divisions. Thus, the second figure, 5, indicates that our number 157 lies between the 5th and 6th secondary divisions as shown in Fig. 1–4.

Third. Read the third figure, which will locate the number on the subdivisions. Thus, the third figure, 7, indicates that our number 157

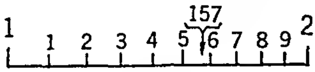

Fig. 1–4.

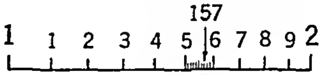

Fig. 1–5.

lies on the 7th subdivision between the 5th and 6th secondary divisions, as shown in Fig. 1–5.

A scale that is complete for the main divisions from 1 to 5 is shown in Fig. 1–6, with the numbers 146, 354, and 465 located thereon. The continuation of the scale from 5 through 6, 7, 8, 9, and 1 would show the same secondary divisions and subdivisions as between 4 and 5.

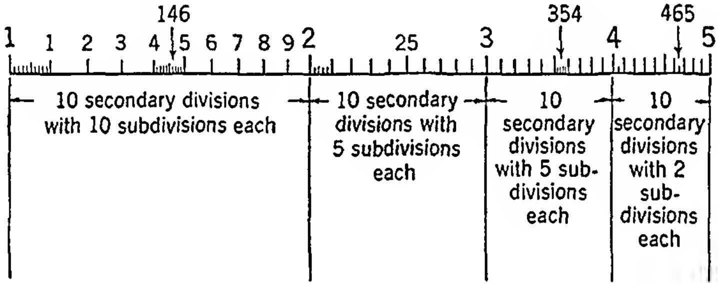

Fig. 1–6.

It is also possible to obtain a 4-digit number by estimating the location of the 4th numeral between the subdivisions. Thus, the number 4,725 is shown in Fig. 1–7.

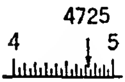

Fig. 1–7.

Numbers that contain more than 4 digits can be set only to the 3rd or 4th place on the rule, but this should cause no difficulty since the percentage of error is so small under such conditions as to be negligible. Thus, the number 175,575 would be set as 175,600. The error introduced here would be 25 in 175,575 or 0.014 per cent, which is small enough so that it can be disregarded.

Now to condense all that has gone before, we can say that:

1. A single-digit number is located at the main divisions.

2. A 2-digit number is located at the secondary divisions.

3. A 3-digit number is located at the subdivisions if the subdivisions are shown. Otherwise its location for the 3rd numeral is estimated as closely as possible.

4. A 4-digit number must be located by estimating the 4th numeral and sometimes by estimating the 3rd numeral.

In locating a number the student should use the hairline on the glass indicator to aid him in following each step. He should also practice setting and reading numbers until he is sure he can do so accurately. Then he is ready to attempt to solve simple problems in multiplication and division.

3. Multiplication of two factors. The rule for the multiplication of two factors is as follows:

Set the index of the C scale (either the left or right index) over one of the factors on the D scale. Run the glass indicator along until the hairline is directly over the second factor on the C scale. Then read the answer on the D scale underneath this second factor on the C. If the slide extends too far to the right when the left index is used, then the right index must be used.

EXAMPLE 1–1. 2×4.

Solution: Opposite 2 on the D scale set the left index of the C scale. Then move the glass runner until the hairline is over 4 on the C scale. Directly underneath this 4 of the C scale will be found 8 on the D scale, which is the answer.

The same result would be obtained by first setting the left index of the C scale opposite 4 on the D scale and then moving the glass runner until the hairline is over 2 on the C scale. Directly below this 2 of the C scale will be found 8 on the D scale, the same answer as before.

EXAMPLE 1–2. 3×4.

Solution: If the left index on the C scale is set opposite the 3 on the D scale, and the glass indicator is moved so as to place the hairline over the 4 on the C scale, it will be found that the hairline extends past the end of the rule and no answer can be read on the D scale.

Therefore, to solve this problem the right index of the C scale must be set opposite 3 on the D scale. Then the glass indicator is moved until the hairline lines up with 4 on the C scale. The answer, 12, will be found directly underneath on the D scale. The same result would have been obtained had the right index of the C been set opposite 4 on D and the hairline moved to line up with 3 on C.

In order to determine which index of the C scale is to be used, the following rule is helpful: *If the product of the first figures of the given numbers is less than 10, the left index should be used; if this product is larger than 10, the right index should be used.*

4. The decimal point. In the two problems that have been discussed, no notice has been taken of the location of the decimal point in the answer, since it has been obvious at a glance. However, in most problems the location of the decimal point is not apparent at once and therefore it must be determined. In the majority of cases the correct position of the decimal point can be determined by putting the figures in round numbers and approximating the result. This will show how many places there must

be to the left of the decimal point in the answer. Rounding off a number to n significant figures means writing the number with only these n significant figures. Thus, the number 528.36 rounded off to four figures is 528.4. Since the figure 6, which is being rejected, is more than ½ unit, the last figure, 3, that is retained, is increased by 1. If 528.36 were to be rounded off to three significant figures, it would be 528. Since the figures being rejected make less than ½ unit, the last figure, 8, that is retained, is kept the same. Rounding off 528.36 to one significant figure would give 500, because the figures rejected do not make more than ½ unit.

In determining the decimal point in an answer to a problem worked by the slide rule, it is generally permissible to round the numbers off to one significant figure, as an approximation is all that is necessary to determine the location of the decimal point. As illustrations: 728 in round figures would be 700; 2.34 would be 2; 51.85 would be 50; 879 would be 900; 3.87 would be 4; and 59.9 would be 60.

EXAMPLE 1–3. 32.2×2.44.

Solution: Making the settings in the regular way, the answer will be found to read 786. By putting those numbers in the round figures of 30 and 2, the answer should be approximately 60 and should have two places to the left of the decimal point. Therefore, the exact answer must be 78.6, rather than 7.86 or 786.

Another easy way to determine the location of the decimal point is to write all numbers in scientific notation. (Refer to Section 32, Chapter 2, Part I.) Thus Example 1–3 might be written $(3.22 \times 10^1)(2.44)$ which gives as an answer (7.86×10^1) or 78.6.

EXERCISE 1–1

Perform the indicated multiplications in the following problems:

1. 2×3.
2. 2.5×3.4.
3. 56×31.
4. 0.425×22.8.
5. 927.5×53.2.
6. $0.025 \times 3{,}406$.
7. 3.96×0.00041.
8. $955 \times 4{,}375$.
9. 6.825×0.7854.
10. 3.1416×0.00625.

5. Multiplication of three or more factors. The multiplication of three or more factors is no more complicated than the multiplication of two.

The first two factors are multiplied together as in the previous cases. It is not necessary to take note of this product, because we are interested in the final product only. Therefore, we shall move the index of the C scale to the product of the first two factors on the D scale, then move the glass indicator until the hairline is over the third factor on the C scale. The answer will be found directly beneath the hairline on the D scale.

This procedure can be followed for any number of factors. Thus, for four factors, first multiply any two of them together in the usual way.

Then multiply their product by one of the remaining factors and finally multiply the product of these three by the final factor.

EXAMPLE 1-4. $3.6 \times 2.28 \times 4.25$.

Solution: Set the left index of the C scale to 3.6 on the D scale. Move the glass indicator until the hairline is over 2.28 on the C scale. The product of these two will be directly underneath the hairline on the D scale. Then leaving the hairline where it is, move the slider until the C index (the right index this time) is directly under the hairline or over the product of the first two factors. Now, move the glass indicator until the hairline is over the third factor, 4.25, on the C scale. The final answer, 349, will be found directly underneath on the D scale. To determine the decimal point, put each one of the three factors in the round numbers, 4, 2, and 4 respectively. Since the product of these round numbers is 32, then our answer must be 34.9.

EXERCISE 1-2

Perform the following indicated multiplications:

1. $376 \times 259 \times 61.2$.
2. $14.65 \times 183.5 \times 2.78$.
3. $0.95 \times 48.7 \times 0.00342$.
4. $2.61 \times 0.057 \times 842$.
5. $279 \times 6.18 \times 0.00218$.
6. $675 \times 321 \times 8{,}625$.
7. $0.0048 \times 0.00072 \times 1.414$.
8. $525 \times 19.05 \times 7.75$.
9. $186 \times 1.732 \times 852$.
10. $2.45 \times 1.414 \times 475$.

6. Division of two factors. Since division is the reverse of multiplication, we can refer back to Example 1-1, in which we multiplied 2 by 4. The very same setting shows 8 divided by 4 to be 2. That is, the divisor is found on the C scale and set directly over the dividend, which is located on the D scale. The answer is found on the D scale underneath the index of the C.

EXAMPLE 1-5. $750 \div 35$.

Solution: Set the divisor 35 on the C scale directly over the dividend 750 on the D scale. The answer will be found to be 21.4 under the index of the C scale. To determine the proper location of the decimal point it is advisable to substitute round numbers in the problem so as to arrive at an approximate answer in which the location of the decimal point is apparent. Thus, if we substitute 700 for 750 and 30 for 35, we see at once that $\frac{750}{30} = 25$. Therefore, the answer we read on the rule must have two digits and is 21.4 rather than 214 or 2.14.

EXAMPLE 1-6. $0.006825 \div 0.025$.

Solution: Set 0.025 on the C scale over 0.006825 on the D scale and read the answer, 273, on D under the index of C. To determine the decimal point, multiply both denominator and numerator by 1,000, making the problem 6.825/25. Now it is obvious that the answer must be 0.273, since $\frac{6}{25}$ is about $\frac{1}{4}$ or 0.25.

EXERCISE 1-3

Perform the indicated divisions in the following problems:

1. $6{,}783 \div 274$.
2. $0.9235 \div 148$.
3. $175{,}269 \div 0.002738$.
4. $0.00875 \div 0.000333$.
5. $0.542 \div 2.42$.
6. $613 \div 0.465$.
7. $0.00652 \div 0.00038$.
8. $0.0395 \div 0.125$.
9. $755 \div 0.06825$.
10. $4.14 \div 85$.

7. Problems involving both multiplication and division. In general, problems that involve both multiplication and division are more numerous than those with only multiplication or only division. In solving such problems it is not necessary to read the answer for each step because we are interested in the final answer only. The best method to follow is to perform division and multiplication alternately, beginning with division. Thus, in the problem $\frac{241 \times 762}{27 \times 143}$, 241 would be divided by 27 as the first step, then this result multiplied by 762, and this result in turn divided by 143. To determine the decimal point, all numbers are put in round figures and an approximation is made. This will determine the number of digits in the answer and therefore the location of the decimal point.

EXAMPLE 1-7. $\frac{83.4 \times 3.14 \times 22.5}{51.25 \times 746}$.

Solution: Step 1—Division: Set 51.25 on C over 83.4 on D.

Step 2—Multiplication: Move the hairline to 3.14 on C.

Step 3—Division: Move 746 on C to the hairline.

Step 4—Multiplication: Move the hairline to 2.25 on C.

Step 5—Read the answer (0.1539) on D directly underneath 2.25 on C.

The decimal point is determined as before by putting all numbers in round figures.

EXAMPLE 1-8. $\frac{81.17 \times 683.5}{47.28 \times 9.65}$.

Solution: Step 1—Division: Set 47.28 on C over 81.17 on D.

Step 2—Multiplication: If the hairline is moved to 683.5 on C, it will be found to run off the end of the scale. Therefore, move the hairline over to the left index of C and then move the slider until the right index of C is under the hairline. This will replace the left index of C with its right index. Now the hairline can be moved to 683.5 on C.

Step 3—Division: Move the slider until 9.65 on C is under the hairline.

Step 4—Read the answer, 121.7, on D directly under the index of C.

Examples 1-7 and 1-8 might also be put into scientific notation form to determine the location of the decimal point more easily. For information on scientific notation see Section 32, Chapter 2, Part I. Thus, Example 1-7 would be:

$$\frac{(8.34 \times 10^1)(3.14)(2.25 \times 10^1)}{(5.125 \times 10^1)(7.46 \times 10^2)} = 1.539 \times 10^{-1} = 0.1539$$

and Example 1–8 would be:

$$\frac{(8.117 \times 10^1)(6.835 \times 10^2)}{(4.728 \times 10^1)(9.65)} = 1.217 \times 10^2 = 121.7.$$

EXERCISE 1–4

Perform all indicated multiplications and divisions in the following:

1. $\dfrac{(1.257)(2.75)(1{,}300)}{(9.82)(25)}$.
2. $\dfrac{(8{,}572.6)(2{,}408.1)}{(66.67)(0.4782)}$.
3. $\dfrac{(3.1416)(1.58)}{(27.2)(0.746)}$.
4. $\dfrac{(0.376)(572)(18.43)(10.15)}{(775)(3.67)(84.3)(29)}$.
5. $\dfrac{(30.3)(835)(50.2)}{(382)(3.74)(41.5)}$.
6. $\dfrac{(23.75)(46.2)(2.86)}{2{,}680}$.
7. $\dfrac{(26.7)(572)(8.7)}{(8{,}360)(2.25)}$.
8. $\dfrac{(22.4)(2.65)(640)}{(0.75)(0.009)(832)}$.
9. $\dfrac{(0.178)(5162)}{(126)(12.4)}$.
10. $\dfrac{(138)(2.78)(625)}{(0.56)(0.006)(4{,}000)}$.

8. Squares and square roots. Now let us turn to the A and B scales. It is apparent at once that they are double scales; that is, each scale is divided into two parts exactly alike and equal, with a left index, a center index, and a right index. Thus, each part makes a complete scale in itself. With the left index chosen as unity or 1, the center index is 10 and the right index is 100; if the left index is chosen as 100, then the center index is 1,000 and the right index 10,000; and so on.

Multiplication and division can be performed with these A and B scales in the same manner as with the C and D scales, but the solution will not have so high a percentage of accuracy, owing to the scales' being half as long. However, they are used primarily for obtaining squares and square roots.

The square of a number is the product found by multiplying the number by itself. Thus, since $3 \times 3 = 9$, then 9 is the square of 3. In symbol form, it is written $3^2 = 9$, which is read, "3 squared equals 9." The square root of a number is one of the two equal factors which when multiplied together will give the number. Thus, since $4 \times 4 = 16$, 4 is the square root of 16. It is written in symbol form: $\sqrt{16} = 4$.

To find the square of a number, set the hairline over the number on the D scale and read the answer on the A scale directly above it. The location of the decimal point can be determined by putting the number in round figures and squaring mentally.

EXAMPLE 1–9. Find the square of 55.8.

Solution: Set the hairline over 558 on the D scale and read the answer, 311, directly above on the A scale. Now, since 50 squared is 2,500, then 55.8 squared must also have four places to the left of the decimal point and will be 3,110, not 311 or 31.1.

To find the square root of a number, the process is reversed. Therefore, *set the hairline over the number on the A scale and read its square root*

directly underneath on the D scale. Some difficulty may arise concerning which part of the A scale to use for the setting of the original number. To overcome this difficulty, begin at the decimal point and mark off the number in groups of two digits each in both directions. If the first group reading from left to right has one digit, then use the left-hand side of the A scale; if the first group has two digits, then use the right-hand side of the A scale. To determine the decimal point in the answer, there must be one digit for each group to the left of the decimal point and one digit for each group to the right of the decimal point in the original number. The following chart may serve to bring out these points more clearly.

Left Index	*Center Index*	*Right Index*
0.000001	0.00001	0.0001
0.0001	0.001	0.01
0.01	0.10	1.00
1.00	10	100
100	1,000	10,000
10,000	100,000	1,000,000

As an illustration, in the first line of figures any number between 0.000001 and 0.00001 will be set on the left side of the A scale to determine its square root, while any number between 0.00001 and 0.0001 will be set on the right side. Likewise, in the fourth line, any number between 1 and 10 will be set on the left side of the A scale to determine its square root, and any number between 10 and 100 will be set on the right side.

EXAMPLE 1–10. $\sqrt{4'38.26'70} = 20.9.$

Solution: Since there is only the one digit, 4, in the first group, set the hairline over 438.267 on the left side of the A scale, and read the answer, 20.9, on the D scale directly under the hairline.

EXAMPLE 1–11. $\sqrt{43'82.67} = 66.2.$

Solution: Since there are two digits, 43, in the first group, set the hairline over 4,382.67 on the right side of the A scale, and read the answer, 66.2, on the D scale directly under the hairline.

EXAMPLE 1–12. $\sqrt{0.07'85} = 0.28.$

Solution: Since there is one digit, 7, in the first group, set the hairline over 785 on the left side of the A scale, and read the answer, 0.28, directly underneath on the D scale.

EXAMPLE 1–13. $\sqrt{0.00'78'50} = 0.0886.$

Solution: Since there are the two digits, 78, in the first group containing figures other than zero, set the hairline over 785 on the right side of the A scale and read the answer, 0.0886, directly underneath on the D scale. Note that one zero has been used in the answer for the group of two zeros at the beginning of the original number.

EXERCISE 1–5

Find the roots indicated in the following:

1. $\sqrt{98{,}265.}$
2. $\sqrt{45.892.}$
3. $\sqrt{0.062178.}$
4. $\sqrt{0.00'00'297.}$
5. $\sqrt{3.1416.}$
6. $\sqrt[4]{7{,}539.61.}$
(*Hint:* Take a square root twice.)
7. $\sqrt{\dfrac{(20.4)(382)}{(4.6)(1.75)}}.$
8. $\sqrt{\dfrac{(0.0001605)(0.00028)}{(0.0001)(0.0205)}}.$
9. $\sqrt{\dfrac{(56.25)(0.00412)}{(1{,}925)(8.560)}}.$
10. $\sqrt[4]{\dfrac{(2.78)(456)}{(0.035)(8.6)}}.$

9. **Cubes and cube roots.** Most slide rules will have a K scale which is used for determining cubes and cube roots. This scale is a triple scale; that is, it is divided into three parts, exactly alike and equal, with an index at the left, an index one third the way to the right, an index two thirds the way to the right, and an index at the extreme right. It therefore has a complete scale for the left third, one for the middle third, and one for the right third.

The cube of a number is the product found by multiplying a number by itself twice. Thus, since $3 \times 3 \times 3 = 27$, then 27 is the cube of 3. In symbol form, it is written $3^3 = 27$ and is read, "3 cubed equals 27."

The cube root of a number is one of the three equal factors which, when multiplied together, will give the number. Thus, since $5 \times 5 \times 5 = 125$, 5 is the cube root of 125. In symbol form it is written $\sqrt[3]{125} = 5$. Note that to indicate the root as a cube root, a small 3 is placed in the $\sqrt{\ }$ of the root indicator. If no number is placed in this $\sqrt{\ }$, the square root is taken; otherwise the root must be indicated.

To find the cube of a number, the hairline is set over the number on the D scale and the answer read directly underneath on the K scale. The decimal point in the answer can be determined by approximation.

Example 1–14. Find $(27.3)^3$.

Solution: Set the hairline over 27.3 on the D scale and read 2,035 on the K scale. To determine the decimal point, find the cube of 30, which gives 27,000 as an approximation. Our answer therefore must have five places and will be 20,350.

To find the cube root of a number, the process is reversed. Therefore, set the hairline at the number on the K scale and read the cube root on the D. However, as in the case of square root, the question arises as to which third of the K scale to use. In order to answer this question, begin at the decimal point and mark off the number in groups of 3 digits each in both directions. If the first group, reading from left to right, has one digit, then use the left third of the K scale; if the first group has two digits, then

use the middle third of the scale; but if the first group has three digits, then use the right third of the scale. To determine the decimal point in the result, there must be one digit for each group to the left of the decimal point and one digit for each group to the right of the decimal point in the original number. The chart herewith will serve to bring these points out more clearly.

Left Index	*One-Third Index*	*Two-Thirds Index*	*Right Index*
0.000001	0.00001	0.0001	0.001
0.001	0.01	0.1	1.00
1.00	10	100	1,000
1,000	10,000	100,000	1,000,000
1,000,000	10,000,000	100,000,000	1,000,000,000

Some slide rules are not provided with a K scale, and this necessitates other methods for finding the cube and the cube root. The cube of a number is easily determined by using the C and D scales and multiplying the number by itself twice, which, in effect, is using the number three times as a factor. Thus, to find 5^3, change it to $5 \times 5 \times 5$ and multiply, as in ordinary multiplication, on the C and D scales. The answer of course comes out to be 125. However, for the cube root a different procedure is required.

To find the cube root of a number, the number is first marked off in groups of three digits as was done for the use of the K scale. This will determine the number of digits in the result, which will have one digit for each group of three on either side of the decimal point. It also permits a mental approximation of the cube root of the first set of digits. Then the hairline is set over the number on the A scale. Next the slider is moved until a number on the B scale under the hairline is exactly the same as a number on the D scale under the index of C. This last number is the cube root of the original number set on the A scale. To determine the side of the A scale to use and the index of the C scale to use for a given number the following rules should be noted:

1. If the first group has one digit, use the left-hand side of the A scale and the left index of C.

2. If the first group has two digits, use the right-hand side of the A scale and the left index of C.

3. If the first group has three digits, use the right-hand side of the A scale and the right index of C.

If it is desired to find the cube root of a problem that has multiplication or division or a combination of both under the radical sign, the A and B scales can be used in exactly the same manner as the C and D scales to perform the multiplications and divisions. If sufficient care is used in

the selection of the proper sections of the A and B scales, the answer under the radical sign will come out on the correct side of the A scale so as to apply the rules for determining the cube root. This procedure will require fewer settings on the rule than using the C and D scales but the accuracy will be reduced to some extent. An approximation of the result under the radical sign will show where this value should come on the A scale in order to get its cube root, and this in turn will determine the proper halves of the A and B scales to be used.

EXAMPLE 1–15. Find $\sqrt[3]{8}$.

Solution: With only one digit to the left of the decimal point, set the indicator to 8 on the left third of the K scale and read 2 on the D scale under the hairline.

To perform this without a K scale, set the hairline over 8 on the left-hand side of the A scale in accordance with Rule *1*, and move the slider until the same number appears on the B scale under the hairline as appears under the left index of C on D. This number will be found to be 2, the same answer as found by use of the K scale.

EXAMPLE 1–16. Find $\sqrt[3]{74'088'000}$.

Solution: Marking off 74,088 in groups of three digits to the left of the decimal point will give two digits, 74, in the first group reading from left to right. The indicator is, therefore, set over 74,088 on the middle third of the K scale and the answer, 42, read on the D scale. The answer actually will be 420, since there are three groups of digits in the original number.

Without a K scale, apply Rule *2*, and the answer, 42, will appear on B under the hairline and on D under the left index of C. The answer of course is 420.

EXAMPLE 1–17. Find $\sqrt[3]{\frac{(182)(496)}{(45)(87)}}$.

Solution: An approximation of the result under the radical sign will give

$$\frac{(182)(496)}{(45)(87)} = 22.$$

Thus, the number whose cube root is desired is approximately 22 and should have two digits. By applying Rule *2*, this should be set on the right-hand side of the A scale. Therefore, we shall use the right-hand side of the A and B scales to perform the multiplications and divisions under the radical sign. This will result in 23.1, whose cube root is desired and, by operating the slider in accordance with Rule *2*, the complete answer is found to be 2.85. So

$$\sqrt[3]{\frac{(182)(496)}{(45)(87)}} = 2.85.$$

EXERCISE 1-6

Find the roots indicated in the following problems:

1. $\sqrt[3]{64}$.
2. $\sqrt[3]{754}$.
3. $\sqrt[3]{3.87}$.
4. $\sqrt[3]{(2.63)(1{,}275)}$.
5. $\sqrt[3]{\dfrac{(21.5)(12.4)}{(2{,}850)(705)}}$.

10. Other scales. The five scales, A, B, C, D, and K, that have been discussed thus far are the more important ones on the slide rule because of their extensive use. But there are several other scales that will be found on most rules. These include a CI scale, a scale of logarithms, a scale for sines of angles, and a scale for tangents of angles.

11. The CI scale. The CI scale is exactly like the C scale with the exception that it extends from right to left instead of from left to right, and therefore it is an inverse scale (CI means C inverse). Because it is an inverse scale, numbers on it are the reciprocals of adjoining numbers on the C scale, and division is performed exactly like multiplication by using the CI and D scales. Thus to perform the division $4/7$, multiply 4 on the D scale by 7 on the CI scale and get the equivalent of $4 \times 1/7$. Its usefulness becomes apparent in a problem involving several multiplications and divisions, where using it to perform the divisions may save several settings on the rule.

12. The sine and tangent scales. A scale (labeled S) of the sines of angles, and a scale (labeled T) of the tangents of angles will be found on the underside of the slider. By reversing the slider and setting it so that the S scale lines up with the A scale and the T scale lines up with the D scale, the values for the sines and tangents of angles can be read directly off the A and D scales respectively. The values of the sine are given up to 90° but the values of the tangent are given up to 45° only. If the tangent of an angle larger than 45° is desired, the reciprocal of the tangent of its complementary angle is found. Thus, if the tangent of 50° is wanted, the tangent of the complementary angle, 40°, is first obtained, and is found to be 0.839. The reciprocal of 0.839, or 1/0.839, is found to be 1.192, and this value is the tangent of 50°.

Caution. When reading the sines of angles, the following should be kept in mind:

All values of the sine that are read on the left half of the A scale will have one zero to the left of the first significant figure and to the right of the decimal point. All values of the sine that are read on the right half of the A scale will have the decimal point just before the first significant figure.

All values of the tangent that are read on the D scale will have the decimal point before the first significant figure.

The values for the sines of angles also can be found by leaving the slider in its original position with the B and C scales showing. In this case a

hairline on the underside of the rule and at one end is used. The desired angle is set opposite this hairline and the sine of the angle is read on the B scale underneath the right index of A. In a similar manner this hairline on the underside of the rule may be used to find the tangents of angles. The desired angle is set under the hairline and the tangent is read on the C scale directly above the right index of D. If the tangent of an angle larger than 45° is required, the reciprocal of the tangent of its complementary angle is found as before.

It will be noted that angles below 5° 43′ cannot be read on the T scale. However, for all practical purposes, the sines and tangents of angles less than 5° 43′ will be the same. Therefore, for such angles the S and A scales can be used to find the tangent.

EXAMPLE 1–18. Find sine 25°.

Solution: Lining up the A and S scales, the value of sine 25° from the A scale is 4,225. Since this reading is on the right side of the A scale, the decimal point will be just before the first significant figure, and therefore sine 25° = 0.4225.

EXAMPLE 1–19. Find sine 3° 40′.

Solution: With the scales lined up, the reading on the A scale directly over 3° 40′ on the S scale is found to be 64. Since this reading is on the left side of the A scale, there will be one zero between the decimal point and the first significant figure. Therefore, the value for sine 3° 40′ is 0.064.

In Examples 1–18 and 1–19, the angles might have been set to the hairline on the underside of the rule and the answers read on the B scale underneath the right index of A.

EXAMPLE 1–20. Find tangent 25° 30′.

Solution: With the T and D scales lined up, the reading on the D scale opposite 25° 30′ is 0.477. (Note that the T scale is marked between 45° and 20° for every 10 minutes, and between 20° and 6° for every 5 minutes.)

EXAMPLE 1–21. Find tangent 67° 30′.

Solution: Since this angle is larger than 45°, we must find the reciprocal of the tangent of its complementary angle. Thus

$$\text{complementary angle} = 90° - 67°\,30' = 22°\,30',$$
$$\text{tangent } 22°\,30' = 0.414,$$
$$\tan 67°\,30' = \frac{1}{\tan 22°\,30'} = \frac{1}{0.414} = 2.414.$$

In Examples 1–20 and 1–21, the angles might have been set to the hairline on the underside of the rule, and the answers read on the D scale directly opposite the right index of C.

13. The logarithm scale. Also on the underside of the slider will be found a scale (labeled L) of the logarithms of numbers. These are loga-

rithms to the base 10. By reversing the slider and lining up the L scale with the D scale, the logarithm of a number can be read on the L scale directly above the number on the D scale, but it must be remembered that the mantissa, or decimal part of the logarithm, is the only part shown on the rule. The characteristic must be supplied by the student.

It is not necessary to reverse the slider if there is a hairline on the reverse side of the rule, as this hairline can be used to determine the logarithm of a number.

If the logarithm scale extends from left to right, the number whose logarithm is desired is set over the right index of D and the logarithm is read on the L scale opposite the hairline on the back.

If the logarithm scale extends from right to left, the left index of C is set over the number on D and the logarithm is read on the L scale opposite the hairline on the back.

14. Other sine and tangent scales. Some slide rules are constructed with a D scale on each side, thereby making it unnecessary to reverse the slider when the sines and tangents of angles are desired. On such rules the S (sine) and T (tangent) scales are used in conjunction with either the D scale or the C scale.

The S scale is used to find the value of the sine or cosine of any angle between 5.7° and 90°, with the same graduations being used for both sines and cosines because the sine of any angle is equal to the cosine of its complementary angle. Therefore two angles are given at each of the longer graduations. On some slide rules the angles may be divided into decimal parts of degrees instead of into minutes and seconds. Thus, the angle 14.4° will be found at the 4th small graduation to the right of the long graduation marked $^{76}\!/_{14}$.

To find the sine of any angle between 5.7° and 90°, line up the index of the S scale with the index of the D scale and set the hairline over the graduation on the S scale that represents the angle. The sine can then be read on either the C scale or the D scale directly under the hairline. The value of the sine or cosine will always be 1 or less and the decimal point for values less than 1 will be at the left of the figures read from the C or D scale; there will be no zeros before the first figure for the sine or cosine of any angle on the S scale. Thus $\sin 14.4° = 0.2487$. This is also $\cos 75.6°$.

The value of the mantissa of log sine of this same angle can also be read on the L scale under the hairline, provided the S and D scales are lined up. Thus, the mantissa of log sine $14.4° = 0.396$. The characteristic, of course, will be determined by the value of the sine and therefore will be -1. Thus, $\log \sin 14.4° = -1.396$. This is also $\log \cos 75.6°$.

The T scale is constructed in a similar fashion to the S scale and is used to determine the tangents and cotangents of angles between 5.7° and 84.3°. By lining up the index of the T scale with the index of the D scale, and setting the hairline over the graduation representing the desired angle, reading on either the C or the D scale will give the tangent of the

hairline on the underside of the rule and at one end is used. The desired angle is set opposite this hairline and the sine of the angle is read on the B scale underneath the right index of A. In a similar manner this hairline on the underside of the rule may be used to find the tangents of angles. The desired angle is set under the hairline and the tangent is read on the C scale directly above the right index of D. If the tangent of an angle larger than 45° is required, the reciprocal of the tangent of its complementary angle is found as before.

It will be noted that angles below 5° 43′ cannot be read on the T scale. However, for all practical purposes, the sines and tangents of angles less than 5° 43′ will be the same. Therefore, for such angles the S and A scales can be used to find the tangent.

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EXAMPLE 1–19. Find sine 3° 40′.

Solution: With the scales lined up, the reading on the A scale directly over 3° 40′ on the S scale is found to be 64. Since this reading is on the left side of the A scale, there will be one zero between the decimal point and the first significant figure. Therefore, the value for sine 3° 40′ is 0.064.

In Examples 1–18 and 1–19, the angles might have been set to the hairline on the underside of the rule and the answers read on the B scale underneath the right index of A.

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$$\tan 67°\,30' = \frac{1}{\tan 22°\,30'} = \frac{1}{0.414} = 2.414.$$

In Examples 1–20 and 1–21, the angles might have been set to the hairline on the underside of the rule, and the answers read on the D scale directly opposite the right index of C.

13. The logarithm scale. Also on the underside of the slider will be found a scale (labeled L) of the logarithms of numbers. These are loga-

rithms to the base 10. By reversing the slider and lining up the L scale with the D scale, the logarithm of a number can be read on the L scale directly above the number on the D scale, but it must be remembered that the mantissa, or decimal part of the logarithm, is the only part shown on the rule. The characteristic must be supplied by the student.

It is not necessary to reverse the slider if there is a hairline on the reverse side of the rule, as this hairline can be used to determine the logarithm of a number.

If the logarithm scale extends from left to right, the number whose logarithm is desired is set over the right index of D and the logarithm is read on the L scale opposite the hairline on the back.

If the logarithm scale extends from right to left, the left index of C is set over the number on D and the logarithm is read on the L scale opposite the hairline on the back.

14. Other sine and tangent scales. Some slide rules are constructed with a D scale on each side, thereby making it unnecessary to reverse the slider when the sines and tangents of angles are desired. On such rules the S (sine) and T (tangent) scales are used in conjunction with either the D scale or the C scale.

The S scale is used to find the value of the sine or cosine of any angle between 5.7° and 90°, with the same graduations being used for both sines and cosines because the sine of any angle is equal to the cosine of its complementary angle. Therefore two angles are given at each of the longer graduations. On some slide rules the angles may be divided into decimal parts of degrees instead of into minutes and seconds. Thus, the angle 14.4° will be found at the 4th small graduation to the right of the long graduation marked $^{76}\!/_{14}$.

To find the sine of any angle between 5.7° and 90°, line up the index of the S scale with the index of the D scale and set the hairline over the graduation on the S scale that represents the angle. The sine can then be read on either the C scale or the D scale directly under the hairline. The value of the sine or cosine will always be 1 or less and the decimal point for values less than 1 will be at the left of the figures read from the C or D scale; there will be no zeros before the first figure for the sine or cosine of any angle on the S scale. Thus sin 14.4° = 0.2487. This is also cos 75.6°.

The value of the mantissa of log sine of this same angle can also be read on the L scale under the hairline, provided the S and D scales are lined up. Thus, the mantissa of log sine 14.4° = 0.396. The characteristic, of course, will be determined by the value of the sine and therefore will be −1. Thus, log sin 14.4° = −1.396. This is also log cos 75.6°.

The T scale is constructed in a similar fashion to the S scale and is used to determine the tangents and cotangents of angles between 5.7° and 84.3°. By lining up the index of the T scale with the index of the D scale, and setting the hairline over the graduation representing the desired angle, reading on either the C or the D scale will give the tangent of the

angle at the right of the graduation and the cotangent of the angle at the left of the graduation. Thus, setting the hairline at 35.6° gives a value of 0.716 on the D scale. This is tan 35.6° and also cot 54.4°. It should be noted that these angles are complementary angles. For the values read on the C or D scales, and when using the T scale for the angle, the decimal point is at the left of the first digit read.

Again, since $\tan \theta = \frac{1}{\cot \theta}$, the CI scale may be used to give the reciprocal value, thereby giving the tangent of the complementary angle. Thus setting the hairline on 35.6° gives a reading of 0.716 on either the C or D scale and also a reading of 1.397 on the CI scale. Therefore from this one setting we find.

$$\tan 35.6^\circ = 0.716 = \cot 54.4^\circ,$$
$$\tan 54.4^\circ = 1.397 = \cot 35.6^\circ.$$

For the values read on the CI scale and when using the T scale for the angle, the decimal point is at the right of the first digit read.

Since the sine and tangent are very nearly equal for small angles, an ST scale has been provided. On this scale, angles less than 5.7° can be set by the hairline and the value of either the sine or tangent can be read on the C or D scale. Sines or tangents of such angles will have one zero before the first digit. The tangent of the complementary angle can be read also on the CI scale with this same setting. Thus, setting the hairline at 3° on the ST scale, we read 542 on the D scale and 191 on the CI scale. Therefore

$$\sin 3^\circ = \cos 87^\circ = \tan 3^\circ = \cot 87^\circ = .0542,$$

and $\tan 87^\circ = \cot 3^\circ = 19.1.$

To multiply a number by the sine or cosine of an angle, set one index of the S or ST scale over the number on the D scale, then move the hairline to the angle on the S or ST scale, and read the answer on the D scale under the hairline. A similar procedure with the T or ST scale and the D scale is followed for multiplication by the tangent of an angle of 45° or less.

To divide a number by the sine or cosine of an angle, set the angle as noted on the S or ST scale over the number on the D scale and read the answer on the D scale opposite the index of the S or ST scale. A similar procedure with the T or ST scale and the D scale is followed for division by the tangent of an angle of 45° or less.

To multiply a number by the tangent of an angle larger than 45°, divide the number by the tangent of the complementary angle.

To divide a number by the tangent of an angle larger than 45°, multiply the number by the tangent of the complementary angle.

SUMMARY

Operation	Scales	Step 1	Step 2	Step 3
To multiply	Use *C* and *D*	Set *index of C to* first number on D	*Move hairline to* second number on C	Read answer under hairline on D
To divide	Use D and C	Set hairline over dividend on D	Find divisor on C and set under hairline	Read answer on D under index of C
To square	Use D and A	Set hairline over number on D	Read answer on A under hairline	
To cube	Use D and K	Set hairline over number on D	Read answer on K opposite hairline	
To find square root	Use A and D Beginning at decimal point, mark number off in groups of 2 digits each in both directions	Set hairline over number on A If first group has 1 digit, use left half of A If first group has 2 digits, use right half of A	Read answer on D under hairline	
To find cube root	Use K and D Beginning at decimal point, mark number off in groups of 3 digits each in both directions	Set hairline over number on K If first group has 1 digit, use left third of K If first group has 2 digits, use middle third of K If first group has 3 digits, use right third of K	Read answer on D under hairline	

EXERCISE 1-7

1. Find sine 36° 30′.
2. Find tangent 32° 20′.
3. Find sine 2° 20′.
4. Find tangent 49° 50′.
5. Find log 2.
6. Find log 30.
7. Find log 5.
8. Find log 17.
9. Find sine 34° 45′.
10. Find tangent 69° 20′.

15. Conclusion. We have now studied the common scales of the slide rule and the methods of using these scales that are in common practice. A condensation of the work in the foregoing sections will be found in the chart on page 19. This chart is a brief summary of the steps necessary when performing the common operations by means of the slide rule.

To acquire confidence in the use of the slide rule and speed in its operation the student will have to practice using his rule wherever and whenever he can. In nothing else is it more true that "practice makes perfect." Therefore, it cannot be stressed too strongly that in order to make the best use of his time the technical student should practice and practice on his slide rule until he can rely upon it completely and perform the operations upon it in a minimum of time and with a maximum degree of accuracy.

We have not concerned ourselves with the theory on which the slide rule is based, since that is more properly taken up after a study of logarithms. Nor have we attempted to introduce any of the special cases wherein the slide rule can be used to good advantage. These will be taken up in their proper places in the further development of mathematical principles. Rather have we confined ourselves to a study of how to make the slide rule go to work for us so that our computations may be made easier and simpler, and may be more quickly done.

EXERCISE 1-8

Perform the operations indicated in the following problems by means of the slide rule.

1. 79×56.
2. $\dfrac{83}{3.8}$.
3. $31 \times 53 \times 0.46$.
4. $\dfrac{26 \times 19}{57}$.
5. $\dfrac{76}{1.3 \times 0.85}$.
6. 0.0347×97.5.
7. $\dfrac{65.3}{0.0081}$.
8. $\dfrac{0.0695 \times 0.5362}{3.658 \times 0.00417}$.
9. $\dfrac{89.74 \times 3.965 \times 672}{3.718 \times 0.0456 \times 0.7854}$.
10. $\sqrt{0.00024863}$.
11. $\sqrt[3]{24{,}863}$.
12. $\sqrt{29{,}876{,}450}$.
13. $\sqrt{298.7645}$.

14. $\frac{392 \times 651 \times 7{,}438}{961 \times 120 \times 537}$.

15. $\frac{4{,}815 \times 9{,}320 \times 0.1356}{48 \times 73.24 \times 1{,}956}$.

16. Fill in the blanks in the following multiplication table:

	23	34	45	56	67
64					
74					
84					
94					

17. Out of 67 problems a student got 62 correct answers. What is his per cent accuracy?

18. A baseball player's batting average is determined by the number of hits he makes, divided by the total number of times at bat. In the season of 1928, Babe Ruth made 173 hits in 536 times at bat. What was his batting average for the year?

In the World Series of that same year, Ruth, in 4 games, made 10 hits in 16 times at bat. What was his batting average for the series?

19. If an airplane traveling 150 mph travels 212 miles in a given time, how far will an automobile averaging 25 mph travel in the same time?

20. The mil is the angle unit in artillery firing and is equal to 1/6,400 of 360 degrees. At 2,400 yd the range of a gun is increased 25.20 yd when the elevation is increased one mil. What change in elevation will increase the range 35 yd?

21. The per cent speed regulation of a motor is the ratio of its change in speed, from no-load to full-load, to the full-load speed; or in equation form:

$$\text{per cent speed regulation} = \frac{\text{no-load speed} - \text{full-load speed}}{\text{full-load speed}} \times 100.$$

(a) If the speed of a motor drops from 1,750 to 1,700 in going from no-load to full-load, what is its speed regulation?

(b) A motor whose no-load speed is 1,800 has a speed regulation of 2.75 per cent. What is its full-load speed?

22. The per cent efficiency of a motor or generator is found by the following equation:

$$\text{per cent efficiency} = \frac{\text{output}}{\text{input}} \times 100.$$

(a) What is the per cent efficiency of a 25-hp (output) motor if the input to the motor is 29.6 hp?

(b) What will be the input to a 15-hp output motor with a per cent efficiency of 82?

23. The effects of wind on a shell are proportional approximately to the velocity of the wind. At 3,000 yd for a 3-in. gun, a rear wind of 10 mph will increase the range 30.1 yd.

(a) What velocity of rear wind will increase the range 41.5 yd?

(b) What velocity of head wind will decrease the range 67.8 yd?

24. The area of a circle is equal to 3.1416 times the square of the diameter divided by 4, or in equation form:

$$A = \frac{(3.1416)(D^2)}{4}.$$

(a) What is the area of a circle with a diameter of 2.5 in.?

(b) What is the diameter of a circle with an area of 6.75 sq. in.?

REVIEW EXERCISE 1–9

Perform the indicated operations in the following:

1. $(8.3)(3.14)$.

2. $(0.0475)(0.00896)$.

3. $(5.03)(6.17)$.

4. $(6716)(0.00185)$.

5. $(4865)(16.5)$.

6. $(7.891)(0.716)(0.395)$.

7. $(605.7)(0.006759)(0.0000172)$.

8. $(5.678)(0.0791)(17500)(83.78)$.

9. $(3.265)(22.17)(90.4)(0.5918)(0.00123)$.

10. $(6.043)(0.000375)(4892)(0.00726)$.

11. $\dfrac{4791}{0.932}$.

12. $\dfrac{6516}{56.1}$.

13. $\dfrac{0.0196}{0.483}$.

14. $\dfrac{105.13}{7.352}$.

15. $\dfrac{0.00475}{348500}$.

16. $\dfrac{(670.6)(4.321)}{(543.8)}$.

17. $\dfrac{(0.00356)}{(3.27)(0.0952)}$.

18. $\dfrac{(4.825)(0.0463)}{725000}$.

19. $\dfrac{(675.1)}{(\pi)(0.00852)}$.

20. $\dfrac{(358.2)(27.16)}{(0.7854)(4.173)}$.

21. $\dfrac{(1.95)(0.1657)(55.4)}{(600)(0.0531)(4.3)}$.

22. $\dfrac{(0.00167)(3048)(0.15)}{(31.4)(6{,}100{,}000)(0.00135)}$.

23. $\dfrac{(0.85)(37.5)}{(6.31)(0.393)(7.005)(0.215)}$.

24. $\dfrac{(36000)(11.7)(34)(2.8)}{(0.073)(342{,}800)}$.

25. $\dfrac{(25)(33{,}000)(0.0156)(2.17)}{(3.14)(1260)(0.375)}$.

26. $\dfrac{(967.5)(0.003265)(0.294)(7.16)}{(8.75)(0.0162)(55{,}500)(0.03875)}$.

27. $\dfrac{(351.67)(8.41)(0.0071)(9.768)}{(12.13)(0.065)(56)(0.17)}$.

28. $\dfrac{(66.17)(3200)(341)(0.0425)}{(0.000872)(625000)}$.

29. $\dfrac{(0.0611)(9300)(38.15)}{(7.82)(3.52)(12750)(0.193)}$.

30. $\dfrac{(53.12)(928)(6.20)(43.16)}{(63000)(72.51)(3.613)(1.392)}$.

31. $\sqrt{0.004835}$.

32. $\sqrt[3]{0.0008751}$.

33. $\sqrt{0.02783}$.

34. $\sqrt[3]{0.05472}$.

35. $\sqrt{275.83}$.

36. $\sqrt{2.7583}$.

37. $\sqrt[3]{2514.8}$.

38. $\sqrt[3]{2.5148}$.
39. $\sqrt{85.724}$.
40. $\sqrt[3]{514.85}$.
41. (4.5)(sin 25.6°).
42. (0.426)(sin 47° 36′).
43. (0.782)(cos 36.8°).
44. (8.31)(cos 72.5°).
45. (2.73)(tan 32.7°).
46. (14.8)(tan 61° 18′).
47. (0.472)(tan 60°).
48. (2.56)(tan 40.5°).
49. $\dfrac{12.73}{\cos 61° \; 12'}$.
50. $\dfrac{0.838}{\cos 26.8°}$.
51. $\dfrac{3.57}{\sin 72.6°}$.
52. $\dfrac{0.0463}{\sin 34.2°}$.
53. $\dfrac{1.578}{\tan 30° \; 24'}$.
54. $\dfrac{3.14}{\tan 52.8°}$.

55. Fill in the blanks in the following multiplication table:

	0.212	3.23	43.4	545.
53.2				
6.43				
0.754				
86.5				
976.				

56. The earned run average for a baseball pitcher is computed by dividing the total number of runs earned against him by the total number of innings he has pitched and multiplying the result by nine. A certain pitcher pitches 287 innings and has 106 earned runs charged against him. What is his earned run average?

57. How many earned runs are charged against a pitcher who has pitched 231 innings and has an earned run average of 2.77?

58. The volume of a sphere is equal to $\frac{4}{3}\pi$ times the cube of the radius or $V = \frac{4}{3}\pi r^3$ where r is the radius.

(a) What is the volume of a sphere with a radius of $1\frac{7}{16}$ in.?
(b) What is the radius of a sphere with a volume of 15.9 cu in.?

59. In a certain type of test, the grade was determined by subtracting the number of incorrect answers from the number of correct answers and dividing by the total number of answers. Out of 225 answers, there were 15 incorrect. What grade was obtained (correct to two places)?

60. The diameter of a circle is equal to $\sqrt{\dfrac{4A}{\pi}}$ where A is the area. Find the diameter of a circle with an area of 28.35 sq in.

61. Determine the area of a circle whose diameter is $2\frac{7}{8}$ in.

62. The area (A) of an equilateral triangle is equal to 0.433 times the square of the base (b) or 0.577 times the square of the altitude (h).

Thus: $A = 0.433b^2$ or $A = 0.577h^2$. Find the area of an equilateral triangle:

(a) with a base of $4\frac{1}{4}$ in.;

(b) with an altitude of $5\frac{3}{4}$ in.

63. Find the base and altitude of an equilateral triangle with an area of 20.25 sq. in.

64. The area of a hexagon is equal to 0.6495 times the square of the distance between opposite corners or 0.866 times the square of the distance between parallel faces. Find the area:

(a) of a hexagon with a distance between opposite corners of $6\frac{1}{8}$ in.;

(b) of a hexagon with a distance between opposite faces of $5\frac{5}{8}$ in.

65. What is the distance between opposite corners of a hexagon whose area is $35\frac{5}{8}$ sq in.? What is the distance between parallel faces of this hexagon?

Chapter 2

ARITHMETIC WITH APPLICATIONS

1. The role of mathematics. Mathematics plays a most important part in the life work of the technically trained man, for it is by means of mathematical statements that he expresses the principles of all the technical advancements that have been and are being formulated. To him it has become second nature to talk and think in terms of mathematics, and this leads him to use the shortest and easiest method in the solution of a problem. He tries to exercise good judgment and always evaluates his answer to see if it is logical. He uses the slide rule, handbooks, and tables to assist him in speeding up his work and in making his calculations concise, clear, and accurate.

2. The method of solution. The form in which a problem is stated and the method of attack in reaching a solution are almost as important as the correct answer. Since most engineering calculations are checked by someone, the checking will be made easy by using the clearest possible form. Oftentimes, the original calculator can find his own mistakes if the work is done clearly and uniformly.

In order to achieve a system of uniformity in the solution of problems, the suggestions that follow will be found helpful.

1. Read the problem carefully, and ascertain what is wanted.

2. If possible, draw a diagram, approximately to scale.

3. Put all known values on the diagram, and be sure they are copied correctly.

4. Work the problem in steps, by setting down the formula to be used in each step and solving the formula for the unknown part in terms of the known values.

5. Substitute the known values in the formulas and solve for the unknown values.

6. Make certain that the units in the problem are all alike. If some weights are given in pounds and others in kilograms, change all of them to the same unit in order to arrive at a correct solution.

7. Do not use calculated results in solving for other unknowns unless absolutely necessary. It is much better to use given data in the solution of each unknown and thereby eliminate a continuation of error made in any one part.

8. Set the problem down in such a form that it can be worked out on a slide rule as far as possible. Omit all long divisions and multiplications; they

tend to obscure and clutter up the solution and should be performed on the slide rule to save time.

9. Include all calculations that are necessary to a clear understanding of the solution.

10. Always use parentheses or brackets to indicate multiplication. Do not use a dot: it may be confused with a decimal point.

3. Significant figures. Significant figures are composed of one or more of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, and the zeros that go to make up a number, provided the zeros are not used in the location of the decimal point. *The zeros at the beginning of a decimal fraction are not significant figures but the zeros at the end of such a fraction are significant figures.* Thus, 25, 0.083, and 1.7 have two significant figures each, while 2,405, 26.82, 0.08761, and 3.140 have four significant figures each.

The zeros at the end of a whole number may be significant or they may be used only to locate the decimal point, in which case they are not significant. The significant figures in a whole number may be indicated by writing the number as a decimal fraction multiplied by some power of 10. Thus, $3,750,000 = 3.75 \times 10^6$ indicates that the number is correct to three significant figures. If it were written as 3.750×10^6, it would be correct to four significant figures. This method of writing a number is called scientific notation and is fully explained in Section 32 of this chapter.

A number is rounded off to n significant figures by writing that number with only n figures. This is done according to the following rules: *The last retained digit is left unchanged when the figures to be rejected represent less than ½ unit of that digit. The last retained digit is increased by 1 when the figures to be rejected represent more than ½ unit of that digit. The last retained digit may be left even or made even if the figures to be rejected represent exactly ½ unit of that digit.* Thus the number 4.31726 is rounded off successively to:

5 significant figures as 4.3173,
4 significant figures as 4.317,
3 significant figures as 4.32,
2 significant figures as 4.3.

The number 6.2415 is rounded off to four significant figures as 6.242, but the number 5.1725 is rounded off to four significant figures as 5.172.

4. Accuracy of results. A very important consideration in the solution of any problem is the accuracy required in the answer. Some problems require greater accuracy than others, but the accuracy of the result is limited by the accuracy of the given data; it cannot be expected that an answer will be more accurate than the data from which it is derived. Thus, if the original data are given accurately to four significant figures, then the answer can be expected to be accurate to four significant figures.

5. General definitions. There are some general definitions in arithmetic that should be reviewed before attempting the solution of problems.

Some of the more important of them follow. Others may be found in standard textbooks on arithmetic.

1. An *integer* or *integral* is a whole number. Examples: 2, 3, 4, 5, 6.
2. A *prime number* is a number that can be divided only by itself and 1. Examples: 1, 3, 5, 7, 11.
3. An *even number* is a number that is exactly divisible by 2. Examples: 2, 4, 16, 28, 40.
4. An *odd number* is a number that is not exactly divisible by 2. Examples: 1, 3, 5, 17, 29.
5. The *minuend* is the number from which another is subtracted.
6. The *subtrahend* is the number that is subtracted.
7. The *remainder* is the result of the subtraction. (It is also that which is left over after a division.)
8. The *multiplicand* is the number to be multiplied by another number.
9. The *multiplier* is the number by which the multiplication is performed.
10. The *product* is the result of the multiplication.
11. The *dividend* is the number to be divided by another number.
12. The *divisor* is the number by which the division is performed.
13. The *quotient* is the result of the division.
14. A *common divisor* or *common factor* of two or more numbers is a number which will exactly divide each of them. If this common factor is the largest one that will exactly divide each of the numbers, it is called the greatest common divisor (gcd).
15. A *multiple* of any number is a number such that the original number will be contained in it an exact whole number of times. The smallest number that will exactly contain each of several numbers is called the least common multiple (lcm).
16. The *reciprocal* of a number is 1 divided by that number. Examples: $\frac{1}{3}$ is the reciprocal of 3; $\frac{1}{8}$ is the reciprocal of 8; 7 is the reciprocal of $\frac{1}{7}$.
17. The part above the dividing line in a division is the *numerator*, whereas the part below the dividing line is the *denominator*. Illustration: In the expression $\frac{2}{3}$,* 2 is the numerator and 3 is the denominator.

6. The fundamental operations. This book assumes that the student has had a sufficient background in arithmetic so that a detailed explanation of the fundamental operations of addition, subtraction, multiplication, and division is not necessary. However, methods for shortening these processes not only are convenient but are time-saving and labor-saving. Therefore, the following short cuts are presented not as a complete list, but to show the possibilities that exist.

* Divisions are expressed also with diagonal dividing line, as $2/3$, with identical meaning. The relative size of the figures has no significance, unless they are exponents in raised, or superior, position. The diagonal is impracticable in complicated equations.

Multiplication:

1. To multiply by 10, add one zero; by 100, add two zeros; by 1,000, add three zeros; and so on.

2. To multiply by 25, add two zeros and divide by 4, because $25 = \frac{100}{4}$.

3. To multiply by 50, add two zeros and divide by 2, because $50 = \frac{100}{2}$.

4. To multiply by 75, add two zeros, divide by 4, and multiply the result by 3, because $75 = \frac{100}{4} \times 3$.

5. To multiply by 125, add three zeros and divide by 8, because $125 = \frac{1{,}000}{8}$.

6. To multiply by 250, add three zeros and divide by 4, because $250 = \frac{1{,}000}{4}$.

7. To multiply by 500, add three zeros and divide by 2, because $500 = \frac{1{,}000}{2}$.

8. To multiply by 750, add three zeros, divide by 4, and multiply the result by 3, because $750 = \frac{1{,}000}{4} \times 3$.

Division:

1. A number whose right-hand digit is 0 or is a figure divisible by 2 may be exactly divided by 2.

2. A number may be exactly divided by 3 if the sum of its digits is divisible by 3.

3. A number may be exactly divided by 4 if the number represented by the last two digits on the right is divisible by 4, or if the original number itself ends in two zeros.

4. A number may be exactly divided by 5 if the last figure on the right is 0 or 5.

5. A number may be exactly divided by 6 if it is an even number and if the sum of its digits is divisible by 3.

6. A number may be exactly divided by 8 if the number represented by the last three digits is divisible by 8.

7. A number may be exactly divided by 9 if the sum of its digits is divisible by 9.

8. Any number ending in zero may be exactly divided by 10.

9. To divide by 125, multiply by 8 and divide by 1,000, because $\frac{1}{125} = \frac{8}{1,000}$.

10. To divide by 250, multiply by 4 and divide by 1,000, because $\frac{1}{250} = \frac{4}{1,000}$.

11. To divide by 500, multiply by 2 and divide by 1,000, because $\frac{1}{500} = \frac{2}{1,000}$.

12. To divide by 750, first divide by 250 as shown in 10 above and then divide this result by 3, because $\frac{1}{750} = \frac{1}{250 \times 3} = \frac{4}{1,000 \times 3}$.

7. Combinations involving addition, subtraction, multiplication, and division. *Arithmetic problems generally are some combination of the four fundamental operations,* and in such cases the importance of the signs cannot be overlooked. The rules that follow are extremely important.

1. A series of additions may be taken in any order. Thus, $9 + 3 + 7 + 5 = 24$ regardless of the order in which the numbers are taken.

2. A series of subtractions should be taken in the order given. Thus, $120 - 25 - 42 = 53$, but if these numbers were taken in some other order the same result might not be obtained.

3. A series of multiplications may be taken in any order. Thus, $5 \times 7 \times 3 = 105$, or $3 \times 5 \times 7 = 105$, or $7 \times 3 \times 5 = 105$.

4. A series of divisions should be taken in the order given. Thus, $150 \div 15 \div 2 = 5$, but if this were taken in some other order the same result might not be obtained.

5. In problems where the four fundamental operations are involved, these operations must be performed in a definite order, if the answer is to have any meaning. A difference of opinion has existed as to what this order should be, but common usage now dictates that the order should be according to the following steps:

Step 1—Perform all multiplications and divisions first in the sequential order in which they are given. (See modification on following page.)

Step 2—Perform additions and subtractions next in either order with respect to each other.

EXAMPLE 2–1. $125 - 50 \div 5 \times 2 + 30$.

Solution: $50 \div 5 = 10$, $\quad 125 - 20 = 105$,
$10 \times 2 = 20$, $\quad 105 + 30 = 135$.

If this problem were to be solved by applying the signs in the order in which they are written, the following would be the result:

$$125 - 50 = 75, \qquad 15 \times 2 = 30,$$
$$75 \div 5 = 15, \qquad 30 + 30 = 60.$$

Since this answer, 60, does not check with the first answer, 135, both of them cannot be correct. The correct answer is 135 because the solution followed the accepted order of doing the multiplications and divisions before the additions and subtractions.

6. In a case where brackets [ ] or parentheses ( ) are used to enclose some portion of the work, the statements in *Step 5* are modified to the extent that the part that is enclosed is performed first. If a number is directly in front of the enclosed portion without any sign intervening, then the result obtained in the enclosure must be multiplied by this number before any other operations are performed.

EXAMPLE 2–2. $750 \div 5 - (30 - 15) + 4(8 - 5)$.

Solution:

$$30 - 15 = 15,$$
$$8 - 5 = 3.$$

Now, rewriting the above, $750 \div 5 - 15 + 4(3)$:

$$750 \div 5 = 150,$$
$$4 \times 3 = 12,$$
$$150 - 15 + 12 = 147.$$

The form in which Example 2–1 is written is not the best one to use because it may cause confusion. A better form to use would be as follows:

$$125 - \left(\frac{50}{5}\right)(2) + 30.$$

This is perfectly clear and leaves no doubt about its solution. Likewise, Example 2–2 would have been better written thus,

$$\frac{750}{5} - (30 - 15) + 4(8 - 5),$$

because it is less confusing.

8. Cancellation. A form like the following example occurs often in the solution of problems.

EXAMPLE 2–3. $\dfrac{(30)(72)(25)(53)}{(5)(55)(42)(24)}$.

This example can be solved by multiplying the numbers in the numerator together to get a single numerator, then multiplying the numbers in the denominator together to get a single denominator and finally dividing the total numerator by the total denominator to get the final result. Such a procedure is long and involved if done by longhand methods, but much of the labor can be eliminated by using the slide rule. However,

it is advisable in either case to simplify the problem first by performing all the divisions possible through a process of cancellation.

Solution:

$$\frac{\overset{\overset{1}{\cancel{6}}}{(\cancel{30})}\overset{3}{(\cancel{72})}\overset{5}{(\cancel{25})}(53)}{\underset{1}{(\cancel{5})}\underset{11}{(\cancel{55})}\underset{7}{(\cancel{42})}\underset{1}{(\cancel{24})}} = \frac{(3)(5)(53)}{(11)(7)} = 10.32.$$

30 divided by 5 gives 6 and 5 divided by 5 gives 1. The 5 is canceled and the figure 1 placed underneath. Also, the 30 is canceled and the figure 6 placed above it.

In like manner, 72 divided by 24 gives 3 and 24 divided by 24 gives 1.

25 and 55 are each divisible by 5, leaving 5 above the 25 and 11 under the 55.

Again, 42 divided by 6 gives 7 and 6 divided by 6 gives 1. The 6 is canceled and 1 placed above it. Also the 42 is canceled and 7 placed underneath.

Finally there is left above the line as a simplified numerator (1)(3)(5)(53), and below the line as a simplified denominator (1)(11)(7)(1). The final result is determined by the slide rule and is found to be 10.32.

This method of cancellation is used when the numbers in the numerator and denominator are multiplied. If addition or subtraction is included in either, then it is better, generally, to perform the operations in the numerator and denominator separately and put the final result down as a fraction which is solved by the slide rule. It is worthy of note that the horizontal line separating the numerator and denominator is equivalent to a bracket or parenthesis or a division sign, and the problem therefore comes under Rule 6 of Article 7.

EXAMPLE 2–4. $\dfrac{(35)(4) + 7 - 9}{(15)(3) + 8 - 5} = \dfrac{138}{48} = \dfrac{23}{8} = 2.875.$

Solution: Simplifying the numerator:

$$(35)(4) + 7 - 9 = 140 + 7 - 9 = 138.$$

Simplifying the denominator:

$$(15)(3) + 8 - 5 = 45 + 8 - 5 = 48.$$

The result, $^{138}\!/_{48}$, can be reduced to its lowest terms by dividing both numerator and denominator by 6, giving $^{23}\!/_{8}$. The final answer will be $2\frac{7}{8}$.

EXERCISE 2-1

Solve the following problems by the rules in Articles 7 and 8. Use cancellation and the slide rule wherever possible.

1. $75 - 32 + (18)(2)$.
2. $127 - (4)(8) + \dfrac{15}{6}$.
3. $\dfrac{315}{9}(6) + 55 - 37$.
4. $\dfrac{1584}{(3)(6)} + (9)(9)$.

5. $\dfrac{81 + 17 - 12 + 4}{(9)(10) - 8 + 18}$.

6. $\dfrac{(957)(3)(28)(225)}{(15)(319)(14)(60)}$.

7. $\dfrac{(72)(85)(7)(841)}{(29)(63)(80)(17)}$.

8. $\dfrac{(15)(72) - (3)(276) + (91)(2)}{(9)(64) + (302)(8) - (6)(27)}$.

9. $\dfrac{825 + 2(16 - 12) - \dfrac{87}{3}}{\dfrac{1470}{7(35 - 28)} - 27}$.

10. $\dfrac{\dfrac{570}{3(28 - 27)} + 5(8 - 3)}{(426)(3) - 4(25 - 15) + 6}$.

11. A rough casting weighs 3 lb 10 oz. After milling and drilling, the finished piece weighs 2 lb 14 oz. The casting costs 24 cents per pound, while the scrap is sold for 8 cents per pound. Find the actual cost of the material for the casting after the piece is finished.

12. Nickel steel has a tensile strength of 90,000 psi of cross section. What pull will a bar ¾ in. on a side be able to withstand? (*Note: psi* is the standard abbreviation for *pounds per square inch.*)

13. In the American wire gauge, No. 4 bare copper wire weighs 126 lb per 1,000 ft. Find the weight and cost of 355 ft if copper is 18 cents per pound.

14. It costs $895,900 per year to purchase coal for a certain power plant. If the coal costs $4.50 per ton, how many tons of coal per day will be used on the average? Assume 365 days in a year.

15. It takes 105 washers of a certain size to make 1 lb. (a) How many full boxes of 12 doz each can be obtained from a keg that weighs 75 lb? (b) How many washers will be left over?

16. One type of sheet steel weighs 0.278 lb per cubic inch. (a) Find the weight of a piece 24 in. × 36 in. × ⅛ in. thick. (b) What will this piece cost at 5 cents per pound?

17. A room is 20 ft long by 15 ft wide and 8 ft high. Find the cost of plastering the walls and ceiling at $1.20 per square yard. Allow 80 sq ft for openings and baseboard.

18. Eight men can unload 1,000 cu yd of gravel in 25 days of 8 hours each. How many men will be needed to do the same unloading in 16 days of 10 hr each?

9. Common fractions. A division such as ⅝ is a fraction. The number above the dividing line is called the *numerator* and the number below the line is called the *denominator*. A common fraction is understood generally to mean a division that does not give a whole number for a result. Examples of such fractions are ⅙, ¾, ⅞.

A fraction whose numerator is smaller than its denominator is called a *proper fraction*, while one whose numerator is larger than its denominator is called an *improper fraction*.

A division that results in a quotient consisting of a whole number and a fraction is called a *mixed number*.

EXAMPLE 2–5. $^{63}/_{5} = 12\,^{3}/_{5}$.

This result means $12 + \,^{3}/_{5}$ and is read "twelve and three fifths." To change the mixed number back to its fractional form, multiply the whole number, 12, by the denominator, 5, of the fraction and add the 3. Place this result over the 5 to complete the fraction. Thus, $(12)(5) + 3 = 63$. Therefore, the fraction is $^{63}/_{5}$.

The following principles should be kept in mind when studying fractions:

1. If two fractions have equal numerators and equal denominators, the fractions are equal.

2. If two fractions have equal denominators, the one with the larger numerator has the greater value.

3. Multiplying or dividing both numerator and denominator of a fraction by the same number does not affect the value of the fraction, because it is equivalent to multiplying the fraction by 1.

4. Multiplying the numerator or dividing the denominator of a fraction by the same number multiplies the fraction by that number. This leads directly to the statement that dividing the denominator by a number is exactly the same as multiplying the numerator by that number.

5. Dividing the numerator or multiplying the denominator of a fraction by the same number divides the fraction by that number. This leads directly to the statement that dividing the numerator by any number is exactly the same as multiplying the denominator by that number.

6. A fraction is in its lowest terms when the numerator and denominator are prime to each other, i.e., *cannot be divided by the same number.* Thus, $^{9}/_{16}$ is in its lowest terms but $^{25}/_{75}$ is not in its lowest terms because both numerator and denominator are divisible by 25. Performing this division gives $^{1}/_{3}$, and this now is the lowest term for the fraction $^{25}/_{75}$.

10. Least common denominator. The smallest number that can be exactly divided by each of the denominators of two or more fractions is the *least common denominator* for those fractions. Thus the least common denominator of $^{1}/_{2}$ and $^{1}/_{3}$ is 6, since 6 is the smallest number that will contain both 2 and 3 exactly.

Similarly, the least common denominator for $^{1}/_{4}$, $^{2}/_{3}$, and $^{1}/_{5}$ is 60, since 60 is the smallest number that exactly contains 4, 3, and 5.

Frequently, it is necessary to change a fraction into one with a larger denominator without changing the value of the fraction. This is done by multiplying both numerator and denominator by the same number, which has been so chosen as to give the new denominator when multiplied by the old denominator. The numerator and denominator of this new fraction are not the same as those of the original fraction, but the fraction itself has the same value.

EXAMPLE 2–6. Change $^{1}/_{3}$ to a fraction with 6 as a denominator.

Solution: Since 3 multiplied by 2 gives 6, the numerator and denominator must each be multiplied by 2. Thus,

$$\frac{1}{3} = \frac{(1)(2)}{(3)(2)} = \frac{2}{6}.$$

In a similar manner:

$$\frac{1}{2} = \frac{(1)(3)}{(2)(3)} = \frac{3}{6} = \frac{(3)(4)}{(6)(4)} = \frac{12}{24} \text{ etc.}$$

$$\frac{1}{5} = \frac{(1)(2)}{(5)(2)} = \frac{2}{10} = \frac{(2)(5)}{(10)(5)} = \frac{10}{50} \text{ etc.}$$

There are occasions when the least common denominator cannot be determined at a glance. In such cases each of the original denominators should be factored into its prime factors. Then, to obtain the least common denominator, each different factor is taken the greatest number of times that it appears in any one denominator and the product of these different factors is determined.

EXAMPLE 2–7. Find the least common denominator of $\frac{1}{16}$, $\frac{1}{20}$, and $\frac{1}{24}$.

Solution: Each of the three denominators is first factored into its prime factors.

$$16 = 2 \times 2 \times 2 \times 2,$$
$$20 = 2 \times 2 \times 5,$$
$$24 = 2 \times 2 \times 2 \times 3.$$

The different factors are 2, 5, and 3. The maximum number of times that 2 appears in any of the above numbers is 4 times in 16. The factors 5 and 3 each appear only once. Therefore, the least common denominator becomes $2 \times 2 \times 2 \times 2 \times 5 \times 3 = 240$.

The fractions are now changed so that each has the least common denominator of 240.

$$\frac{1}{16} = \frac{(1)(15)}{(16)(15)} = \frac{15}{240};$$
$$\frac{1}{20} = \frac{(1)(12)}{(20)(12)} = \frac{12}{240};$$
$$\frac{1}{24} = \frac{(1)(10)}{(24)(10)} = \frac{10}{240}.$$

11. Addition and subtraction of common fractions. To add or subtract fractions, they are all changed so as to have the same least common denominator. Addition is then performed by adding the numerators and placing the sum over the common denominator; subtraction is performed by subtracting the numerators and placing the difference over the common denominator.

EXAMPLE 2–8. Add: $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{5}$.

Solution: The least common denominator is 30.

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{5} = \frac{15}{30} + \frac{20}{30} + \frac{18}{30} = \frac{15 + 20 + 18}{30} = \frac{53}{30} = 1\frac{23}{30}.$$

EXAMPLE 2–9. Subtract $\frac{2}{3}$ from $\frac{7}{8}$.

Solution:
$$\frac{7}{8} - \frac{2}{3} = \frac{21}{24} - \frac{16}{24} = \frac{5}{24}.$$

12. Multiplication and division of common fractions. To multiply two or more fractions together, form a new fraction with a numerator that is equal to the product of the numerators of the original fractions and a denominator that is equal to the product of the denominators of the original fractions.

To multiply a fraction by a whole number, consider the whole number to be a fraction with a denominator of one and multiply as above.

EXAMPLE 2–10. Multiply: $\frac{3}{8} \times \frac{2}{5} \times \frac{7}{9}$.

Solution:
$$\frac{3}{8} \times \frac{2}{5} \times \frac{7}{9} = \frac{(3)(2)(7)}{(8)(5)(9)} = \frac{42}{360} = \frac{7}{60}.$$

Dividing both numerator and denominator by 6 reduces $42/360$ to its lowest terms of $7/60$. To arrive at the form of the lowest answer directly, cancellation might have been used before multiplying. Thus,

$$\frac{3}{8} \times \frac{2}{5} \times \frac{7}{9} = \frac{(\cancel{3})(\cancel{2})(7)}{\underset{4}{(\cancel{8})}(5)\underset{3}{(\cancel{9})}} = \frac{7}{60}.$$

EXAMPLE 2–11. Multiply $\frac{2}{7}$ by 4.

Solution:
$$\frac{2}{7} \times 4 = \frac{2}{7} \times \frac{4}{1} = \frac{(2)(4)}{(7)(1)} = \frac{8}{7} = 1\tfrac{1}{7}.$$

To divide one fraction by another, invert the second fraction (interchange its numerator and denominator) and multiply the first fraction by this inverted form.

To divide a fraction by a whole number, consider this whole number to be a fraction with a denominator of 1 and proceed as above.

EXAMPLE 2–12. Divide $\frac{2}{3}$ by $\frac{5}{7}$.

Solution:
$$\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{2}{3} \times \frac{7}{5} = \frac{(2)(7)}{(3)(5)} = \frac{14}{15}.$$

EXAMPLE 2–13. Divide $\frac{2}{3}$ by 5.

Solution:
$$\frac{\frac{2}{3}}{5} = \frac{2}{3} \times \frac{1}{5} = \frac{(2)(1)}{(3)(5)} = \frac{2}{15}.$$

EXAMPLE 2–14. A copper bar $25\frac{1}{2}$ in. long is to be cut into pieces each $4\frac{3}{4}$ in. long. How many pieces of this length will there be and how much material will be left over? Allow $\frac{1}{32}$ in. for the saw cut.

Solution:

$$\frac{25\frac{1}{2}}{4\frac{3}{4}} = \frac{\frac{51}{2}}{\frac{19}{4}} = \left(\frac{51}{\cancel{2}}\right)\left(\frac{\overset{2}{\cancel{4}}}{19}\right) = \frac{102}{19} = 5\frac{7}{19}.$$

Therefore, there will be 5 pieces each $4\frac{3}{4}$ in. long and a smaller piece left over. The length of the piece left over is determined as follows:

Length of 5 pieces is

$$5 \times 4\frac{3}{4}\text{ in.} = (5)\left(\frac{19}{4}\right) = \frac{95}{4} = 23\frac{3}{4}\text{ in.}$$

Length of 4 saw cuts for 5 pieces is

$$4 \times \frac{1}{32}\text{ in.} = (4)\left(\frac{1}{32}\right) = \frac{4}{32} = \frac{1}{8}\text{ in.}$$

Total length taken out is

$$23\frac{3}{4}\text{ in.} + \frac{1}{8}\text{ in.} = \frac{95}{4} + \frac{1}{8} = \frac{190}{8} + \frac{1}{8} = \frac{191}{8} = 23\frac{7}{8}\text{ in.}$$

Length left over is

$$25\frac{1}{2}\text{ in.} - 23\frac{7}{8}\text{ in.} = 1\frac{5}{8}\text{ in.}$$

13. Addition, subtraction, multiplication, and division of common fractions in combinations. Generally, the problems encountered in fractions will contain some combination of the four fundamental processes rather than being confined to one. In the solution of such combination problems, the rules laid down in Article 7 are followed. Multiplication and division are performed before addition and subtraction unless brackets or parentheses are inserted around some portion of the problem, and in such a case the portions within the brackets or parentheses are performed first.

EXAMPLE 2–15. Solve the following problem:

$$\frac{\frac{22}{18} \times \frac{36}{64} \div \frac{2}{3} - \frac{1}{8}}{\left(\frac{3}{4} + \frac{2}{3}\right)\left(\frac{50}{51}\right) \div \left(\frac{7}{12} - \frac{3}{18}\right)}.$$

The numerator is simplified first:

$$\frac{22}{18} \times \frac{36}{64} \div \frac{2}{3} = \frac{\overset{11}{\cancel{22}}}{\cancel{18}} \times \frac{\overset{\cancel{2}}{\cancel{36}}}{\underset{32}{\cancel{64}}} \times \frac{3}{\cancel{2}} = \frac{(11)(3)}{32} = \frac{33}{32},$$

$$\frac{33}{32} - \frac{1}{8} = \frac{33}{32} - \frac{4}{32} = \frac{29}{32}.$$

The simplified numerator therefore is $\frac{29}{32}$.

Next, the denominator is simplified:

$$\frac{3}{4}+\frac{2}{3}=\frac{9}{12}+\frac{8}{12}=\frac{17}{12},$$

$$\frac{7}{12}-\frac{3}{18}=\frac{21}{36}-\frac{6}{36}=\frac{15}{36},$$

$$\frac{17}{12}\times\frac{50}{51}\div\frac{15}{36}=\frac{\cancel{17}}{\cancel{12}}\times\frac{\overset{10}{\cancel{50}}}{\underset{3}{\cancel{51}}}\times\frac{\overset{\cancel{12}}{\cancel{36}}}{\underset{3}{\cancel{15}}}=\frac{10}{3}.$$

Now the total expression, simplified, is:

$$\frac{\frac{29}{32}}{\frac{10}{3}}$$

and its solution

$$\frac{29}{32}\times\frac{3}{10}=\frac{87}{320}.$$

EXERCISE 2-2

Perform the indicated operations in the following:

1. $\frac{3}{4}+\frac{4}{5}+\frac{5}{6}.$
2. $2\frac{2}{3}-1\frac{1}{6}+3\frac{4}{9}.$
3. $3\frac{3}{8}-\frac{9}{16}.$
4. $10\frac{15}{32}-2\frac{5}{16}.$
5. $2\frac{5}{8}+3\frac{1}{4}-\frac{15}{16}.$
6. $\dfrac{\left(\frac{2}{3}\right)\left(\frac{7}{13}\right)+\frac{17}{39}}{\left(\frac{5}{8}-\frac{1}{4}\right)\left(\frac{8}{13}\right)}.$
7. $\dfrac{\frac{9}{11}-\frac{4}{33}+\frac{5}{66}}{\frac{8}{9}+\frac{2}{3}-\frac{5}{18}}.$
8. $\dfrac{\left(\frac{9}{17}\right)\left(\frac{14}{42}\right)\left(\frac{3}{2}\right)-\frac{1}{2}+\frac{6}{7}}{\dfrac{\left(\frac{3}{4}\right)\left(\frac{17}{18}\right)}{\frac{3}{8}}-\frac{1}{4}+\frac{3}{8}}.$
9. $\dfrac{\left(\frac{5}{9}\right)\left(\frac{2}{3}\right)+\frac{11}{18}-\frac{1}{9}}{\frac{17}{21}+\left(1\frac{7}{9}\right)\left(\frac{4}{7}\right)+\dfrac{\frac{2}{3}-\frac{4}{9}}{\frac{1}{3}-\frac{6}{21}}}.$
10. $\dfrac{\dfrac{\left(\dfrac{\frac{5}{6}}{\frac{1}{3}}\right)\left(\frac{4}{9}\right)\left(\frac{11}{18}\right)}{\left(\dfrac{\frac{1}{2}}{\frac{3}{5}}\right)\left(\frac{5}{9}\right)-\frac{1}{4}}}{\dfrac{\frac{7}{8}+\left(\frac{1}{3}\right)\left(\frac{4}{5}\right)-\frac{1}{6}}{\frac{1}{5}+\frac{3}{4}-\dfrac{\frac{5}{8}}{1\frac{2}{3}}}}.$

11. Continued fractions are found frequently in compound gearing problems. They are solved by starting at the bottom and working toward the top. Solve the following continued problem:

$$2+\cfrac{1}{2-\cfrac{1}{2+\cfrac{1}{2-\cfrac{1}{2+\cfrac{1}{2-\cfrac{1}{2}}}}}}$$

12. How many pieces of aluminum can be cut from a strip $32\frac{3}{4}$ in. long if each piece is $5\frac{1}{2}$ in. long? How much material is left over?

13. A water tank is supplied by two pipes and drained by a third. The first can fill $\frac{1}{3}$ of the tank in 1 hr and the second can fill $\frac{1}{4}$ of the tank in 1 hr. The third can drain $\frac{5}{8}$ of the tank in 1 hr. If all the pipes are open, how long will it take to empty the tank?

14. A brass rod 10 ft long is to be cut into screws each $\frac{3}{4}$ in. long. If $\frac{1}{8}$ in. is wasted at each cut, how many screws can be made?

15. A motor shaft is $2\frac{3}{16}$ in. in diameter and is to be cut down to $1\frac{13}{16}$ in. in diameter. How deep a cut is necessary on the lathe that is used for turning the shaft down to correct size?

16. With a negative size of $2\frac{1}{2}$ in. $\times$ $4\frac{1}{4}$ in., how many prints can be made from a sheet of print paper 36 in. $\times$ 48 in. if $\frac{1}{8}$ in. is to be left for a border and $\frac{1}{16}$ in. all around for trimming?

17. The standard length of a bolt is given as the distance from under the head to the extreme end. A bolt is measured to have the following dimensions: length of head $\frac{25}{64}$ in.; length of thread $1\frac{3}{16}$ in.; length of unthreaded portion $\frac{9}{16}$ in. Find its standard length and its over-all length.

18. A circular metal rod is $15\frac{1}{2}$ in. long and weighs $6\frac{1}{4}$ lb. (a) What is the weight of a piece $2\frac{3}{4}$ in. long cut from the rod? (b) What is the length of a piece that weighs $3\frac{5}{8}$ lb?

19. A lathe runs at a speed of 120 rpm and has a feed of $\frac{3}{64}$ in. per revolution. How long in seconds will it take to turn a rod 18 in. long?

14. Decimal fractions. A decimal fraction is a fraction that has 10 or some power of 10 as a denominator. Examples are:

$$\frac{7}{10}; \frac{15}{100}; \frac{29}{1{,}000}.$$

Such fractions usually are written by omitting the denominator and indicating it by the use of a period (.) or decimal point, in the numerator. The number of places to the right of the decimal point should be the same as the number of zeros in the denominator. Thus:

$$\frac{7}{10} = 0.7; \frac{15}{100} = 0.15; \frac{29}{1{,}000} = 0.029.$$

In the third sample it has been necessary to put a zero at the left of the figure 29 in order to make up three places for the three zeros in the denominator.

It is clear, from the above explanation, that the location of the decimal point is very important. The decimal fraction is multiplied by 10 for each place the decimal point is moved to the right, and is divided by 10 for each place the decimal point is moved to the left. For example 1.45 becomes 14.5 by moving the point one place toward the right and it becomes 0.145 when the point is moved one place toward the left.

A decimal fraction is generally written with a zero preceding the decimal point so that there will be no mistake about the location of the decimal point. Thus, .25 would be written as 0.25.

15. Addition and subtraction of decimals. To add or subtract decimals, write the numbers with their decimal points in a vertical line and then proceed as in any addition or subtraction of whole numbers. Zeros should be added at the right where necessary to give enough decimal places.

EXAMPLE 2–16. Add 1.46, 1.904, 5.0032, and 3.5.

Solution:

$$\begin{array}{r} 1.4600 \\ 1.9040 \\ 5.0032 \\ \underline{3.5000} \\ 11.8672 \end{array}$$

EXAMPLE 2–17. Subtract 3.486 from 7.5912.

Solution:

$$\begin{array}{r} 7.5912 \\ \underline{-\ 3.4860} \\ 4.1052 \end{array}$$

A check for this can be made by adding the remainder and subtrahend to see if the sum gives the minuend.

16. Multiplication of decimals. To multiply any two decimal numbers, disregard the decimal points and multiply the same as with any two whole numbers. Then, locate the decimal point as many places from the right in the result as the sum of the places at the right of the decimal point in the original numbers.

EXAMPLE 2–18. Multiply 2.512 by 43.6.

Solution:

$$\begin{array}{r} 2.512 \\ \underline{43.6} \\ 1\ 5072 \\ 7\ 536 \\ \underline{100\ 48} \\ 109.5232 \end{array}$$

The result has four decimal places because there are three places in one factor and one place in the other, giving a sum of four. The answer

can be checked readily by using approximations for the two original numbers. Thus, (2.512)(43.6) is roughly (3)(44) or about 132. Therefore, the answer must be 109.5232 instead of 10.95232 or 1,095.232.

To multiply a decimal by 10, move the decimal point one place to the right; to multiply by 100, move the decimal point two places to the right, and so on. *In general, to multiply by 10 or any power of 10, move the decimal point as many places to the right as there are zeros in the multiplier.*

17. Division of decimals. To divide by a decimal, the division should first be carried through completely without regard for the decimal points. Then the same number of places should be pointed off in the quotient as shown by the difference of the decimal places in the dividend and divisor. The dividend must always have at least as many decimal places as the divisor, and therefore it may be necessary to add zeros to the dividend to fulfill this condition. Another method that will achieve the same result is to multiply both dividend and divisor by the power of 10 that makes the divisor a whole number. The quotient will then have the same number of places as the new dividend.

To divide a decimal by 10, move the decimal point one place to the left; to divide by 100, move the decimal point two places to the left, and so on. *In general, to divide by 10 or any power of 10, move the decimal point as many places to the left as there are zeros in the divisor.*

EXAMPLE 2–19. Divide 15.312 by 2.64.

Solution:

$$\begin{array}{r@{}l}
 & 5.8 \\
2.64 & \overline{)15.312} \\
 & \underline{13\,20} \\
 & 2\,112 \\
 & \underline{2\,112}
\end{array}$$

Since the dividend has three decimal places and the divisor two places, the difference, one place, is pointed off in the quotient. This might have been done as follows:

$$\begin{array}{r@{}l}
 & 5.8 \\
264. & \overline{)1531.2} \\
 & \underline{1320} \\
 & 211\,2 \\
 & \underline{211\,2}
\end{array}$$

Both dividend and divisor are multiplied by 100, so as to make the divisor a whole number, by moving the decimal points two places to the right. The old decimal points are crossed out and the new decimal points inserted and the work is arranged so that the decimal point in the answer is directly above the new decimal point in the dividend.

The work can be checked for the location of the decimal point by an approximation of the numbers. Thus, $15.312 \div 2.64$ is roughly $15 \div 3$, or 5. Therefore, the quotient must be 5.8 instead of 0.58 or 58.

Since the remainder is the impr+ 3 = 4 residue
digits, the work can be simplified b→ = 6 residue
total 9. Thus for 35,248, the 5
to 9, leaving 3 + 2 + 8 = 13 =
before.

This rule of nine can be appliec
problems as a check to see if the sc

When it is desired to check the
residue or excess of nines is found i
residues should be the same as the + 4 + 8 = 14 = 1 + 4 = 5.
bers. Nines may be cast out wher quotient residue) + remainder residue

EXAMPLE 2–22. Check the foll

(4) + 5

Residue + 5
casting + 3 = 6
nin(

Solution: 3,562 → 7
2,541 → 3
2,439 → 0
6,743 → 11 → 2
15,285 7

1 + 5 + 2 + 8 +

problem by using the residue of
tion of the location of the decimal
e decimal point can be determined
per values to give an approximate
presented in Chapter 1, The Slide

2–3

When it is desired to check mul
found and the product of these r05, and 121.
product of the numbers. Again, idue of nines)
sible.
09.

EXAMPLE 2–23. Multiply 275. idue of nines)
nines.
)5.2.
idue of nines)

Solution: 2 7 5.4 6 → residu 1.241,

1 8.5 3 → residue 1.1458,
8 2 6 3 8 1.14826.
1 3 7 7 3 0 lue of nines:
2 2 0 3 6 8 .00546,
2 7 5 4 6 11.04,
5 1 0 4.2 7 3 8 residue = 3.00426,
.00025.

When it is desired to check divisi of nines:
quotient, and remainder are found
23.5,
checked against the product of the
1.98,
plus the residue of the remainder.
0.00841,

EXAMPLE 2–24. Divide 5,469.0.627.

7. Change to decimals or mixed numbers:

$$\frac{1}{75.5}; \frac{82}{15.6}; \frac{743}{2584}; \frac{10}{256}; \frac{100}{28}.$$

8. Change the following to common fractions:

0.782; 0.95; 1.48; 20.750; 0.0625;
0.482; 0.0085; 0.0254; 25.18.

9. A piece of metal stock is 56.38 in. long. How many pieces each 4.625 in. long can be cut from it if 0.025 in. is wasted at each cut? How much material is left over?

10. A piece of conduit has an outside diameter of 1.315 in. and an inside diameter of 1.049 in. What is the thickness of the wall of the conduit in decimal form and in common-fraction form?

11. A sheet of metal weighs 0.256 lb per square foot. How many square feet will there be in a piece that weighs 175 lb?

12. A cubic foot of water weighs approximately 62.4 lb. (a) How many cubic feet are there in a ton (2,000 lb) of water? (b) How much will a cubic inch of water weigh?

22. Percentage. Decimal fractions that have a denominator of 100 are particularly significant in engineering problems. They are expressed by means of the term *per cent*, which means "by the hundred." Thus, 25 per cent has the same meaning as the decimal fraction 0.25. To change a decimal fraction into per cent form, the fraction is multiplied by 100 and the decimal point is moved two places to the right. Also, the symbol % is used in place of the words "per cent." Thus, 0.25 = 25%, which is read "twenty-five per cent"; and 0.032 = 3.2%, which is read "three and two tenths per cent." The symbol % is a contracted form of the divisor $\frac{}{100}$.

To change a common fraction to per cent, the fraction is changed to a decimal and this decimal is written in terms of per cent. Thus,

$$\frac{3}{4} = \frac{75}{100} = 0.75 = 75\%.$$

It is therefore evident that the sign % does duty for two decimal places.

In problems dealing with percentage, the terms *base, rate,* and *percentage* are used, and their relationship is shown by the expression: *percentage = base times rate.* This expression also can be rewritten so as to solve for the base or solve for the rate:

$$\text{base} = \frac{\text{percentage}}{\text{rate}}, \text{ or rate} = \frac{\text{percentage}}{\text{base}}.$$

It should be noted that the rate is the figure to which the % sign is attached and is therefore a decimal fraction while the base and percentage are simply numbers.

Since the expression that shows the relationship among the base, the rate, and the percentage can be written in three ways, there are three type problems involving this expression.

EXAMPLE 2-25. Find $37\frac{1}{2}\%$ of 240.

Solution: The base is 240, the rate is $37\frac{1}{2}\%$, and the percentage is to be found.

$$\text{percentage} = \text{base times rate},$$
$$37\tfrac{1}{2}\% = 0.375.$$

So
$$\text{percentage} = (240)(0.375) = 90.$$

Therefore
$$37\tfrac{1}{2}\% \text{ of } 240 \text{ is } 90.$$

EXAMPLE 2-26. 18 is what per cent of 60?

Solution: The percentage is 18, the base is 60, and the rate is to be found.

$$\text{rate} = \frac{\text{percentage}}{\text{base}},$$
$$\text{rate} = \frac{18}{60} = \frac{3}{10} = \frac{30}{100} = 0.30 = 30\%.$$

Therefore
$$18 \text{ is } 30\% \text{ of } 60.$$

EXAMPLE 2-27. 32 is 25% of what number?

Solution: The percentage is 32, the rate is 25%, and the base is to be found.

$$\text{base} = \frac{\text{percentage}}{\text{rate}},$$
$$25\% = 0.25,$$
$$\text{base} = \frac{32}{0.25} = 128.$$

Therefore
$$32 \text{ is } 25\% \text{ of } 128.$$

The applications of percentage in the engineering and business fields are numerous and varied. For instance, *the efficiency of a machine is defined as its output divided by its input*, and, since output is always smaller than input, this division results in a decimal which is changed to per cent by multiplying by 100 and attaching the per cent symbol (%). Therefore, it follows that

$$\%\ \text{efficiency} = \frac{\text{output}}{\text{input}} \times 100.$$

It should be remembered that both the output and the input must be expressed in the same units to make the expression valid.

When the efficiencies of parts of a machine or piece of equipment are given, *the combined efficiency is the product of the efficiencies of the parts.* For example, a motor with an efficiency of 80% is driving a pump that has an efficiency of 75%. The over-all efficiency of the set is $0.80 \times 0.75 = 0.60$ or 60%.

The speed regulation of a motor is defined as the difference between the no-load and full-load speeds expressed as a per cent of the full-load speed. Thus,

$$\% \text{ speed regulation} = \frac{\text{no-load speed} - \text{full-load speed}}{\text{full-load speed}} \times 100.$$

The voltage regulation of a transformer is defined as the difference between the no-load and full-load secondary voltages expressed as a per cent of the full-load secondary voltage. Thus,

$$\% \text{ voltage regulation} = \frac{\text{no-load voltage} - \text{full-load voltage}}{\text{full-load voltage}} \times 100.$$

Tax rates are figured on a basis of \$1,000 units of assessed valuation. Thus, \$30 per \$1,000 means a decimal of 0.03 or 3%. A piece of property with an assessed valuation of \$5,000 and a tax rate of 0.03 would be taxed for $0.03 \times 5{,}000$, or 3×50, which would be \$150 per year.

Cost and selling prices are figured in various ways but the per cent usually is based upon some cost such as labor and material, wholesale dealer's price, etc. If several per cents are given, they are applied in turn.

The applications of percentage are limitless; and it would be impossible to include all of them in this work. The following are a few examples of those that affect the technical field.

EXAMPLE 2–28. A motor that is rated to deliver 5 hp output requires an input of 5,000 w. What is the efficiency of the motor?

Solution: The output and input must be in the same units. Therefore, to change 5 hp to watts, multiply by 746.

$$\text{output} = 5 \text{ hp} = 5 \times 746 = 3{,}730 \text{ w},$$

$$\% \text{ efficiency} = \frac{\text{output}}{\text{input}} \times 100,$$

$$\% \text{ efficiency} = \frac{3730}{5000} \times 100 = 75\%.$$

EXAMPLE 2–29. The speed of a motor dropped from 1,750 rpm at no-load to 1,700 rpm at full-load. What is its speed regulation?

Solution:

$$\% \text{ speed regulation} = \frac{\text{no-load speed} - \text{full-load speed}}{\text{full-load speed}} \times 100,$$

$$\% \text{ speed regulation} = \frac{1750 - 1700}{1700} \times 100 = 2.94\%.$$

EXAMPLE 2–30. The secondary voltage rating of a transformer is 110 v at no-load. When full-load is applied this voltage drops to 108.5 v. What is its voltage regulation?

Solution:

$$\%\,\text{voltage regulation} = \frac{\text{no-load voltage} - \text{full-load voltage}}{\text{full-load voltage}} \times 100,$$

$$\%\,\text{voltage regulation} = \left(\frac{110 - 108.5}{108.5} \times 100\right) = \left(\frac{1.5}{108.5} \times 100\right) = 1.383\%.$$

23. Averages. In engineering calculations, the values that are used are often obtained from measurements or observed readings and as a result they are subject to error. To correct this condition as far as possible and obtain an accurate value, several measurements or readings are taken. Then the readings are added together and the sum is divided by the total number of readings. This result is the average of all the readings and is accepted usually as the correct value. It is the average or arithmetic mean of the total set of numbers. Therefore, to generalize, the average or arithmetic mean of a set of m numbers is determined by dividing the sum of the numbers by m. In determining this average it should be kept in mind that the quantities that are added must be of the same kind and must refer to the same unit. As an illustration, the average speed of an automobile over a given route is the total distance traveled divided by the time taken to travel that distance or

$$\text{speed} = \frac{\text{distance}}{\text{time}}.$$

24. Per cent error. It is seen from the preceding article that none of the individual readings or measurements may check exactly with the average value. If the average is accepted as the correct value, then the other values may have some error in them. The error in any particular value is the difference between it and the average, and this is usually expressed in per cent of the average.

EXAMPLE 2–31. The following readings were taken in measuring the diameter of a copper wire at different points along its length: 0.1275 in.; 0.1270 in.; 0.1290 in.; 0.1285 in. Determine the average diameter and the per cent error in the first reading.

Solution: The sum of the readings is 0.1275 in. + 0.1270 in. + 0.1290 in. + 0.1285 in. = 0.5120 in.

$$\text{The average} = \frac{0.5120}{4}\text{ in.} = 0.1280\text{ in.},$$

$$\text{error in first reading} = 0.1280\text{ in.} - 0.1275\text{ in.} = 0.0005\text{ in.},$$

$$\frac{0.0005}{0.1280} = 0.00391 = 0.391\%\text{ error.}$$

EXERCISE 2–4

1. Find

15% of 540,
$62\frac{1}{2}\%$ of 1.424,
$12\frac{1}{2}\%$ of 0.8572,
$87\frac{1}{2}\%$ of 643.24.

2. 2 is what per cent of 11?
25 is what per cent of 70?
18 is what per cent of 12?
87 is what per cent of 36?

3. 615 is 70% of what number?
25 is $12\frac{1}{2}$% of what number?
0.065 is $62\frac{1}{2}$% of what number?
1.75 is 25% of what number?

4. A certain motor is 88.5% efficient. The output of the motor at full-load is 15 hp. What is the input to the motor in watts?

5. The speed regulation for a certain motor is found to be 2.37%. Its no-load speed is 1,750 rpm. What is its full-load speed?

6. One kind of brass is made of the following constituents by weight: 61.6% copper, 2.9% lead, 0.2% tin, and the remainder is zinc. How many pounds of each must be added to 150 lb of copper in making a brass rail?

7. The following readings were taken to determine the correct diameter of a shaft: 2.079 in.; 2.075 in.; 2.078 in.; 2.080 in.; 2.077 in.; 2.081. What was the average diameter and the per cent error in the largest reading?

8. A transformer rated at 120 v on the secondary side at no-load has a voltage regulation of 1.42% when full-load is applied. What is the full-load secondary voltage?

9. At 4,000 ft height, the range of visibility from an airplane is 66 miles. The pilot increased his altitude to 7,000 ft and thereby increased his range of visibility to 80 miles. What was his per cent increase in altitude and his per cent increase in visibility?

10. A casting weighed 47.82 lb before finishing and 45.28 lb after finishing. What per cent of the original weight was removed in the finishing process?

11. A factory employee is being paid 72 cents an hour for 40 hr a week, with time and a half for any time over 40 hr. He receives an increase of 10 cents an hour. What is his per cent increase and what is his total salary at the new rate for a 48-hr week?

12. A finished casting weighed 52.35 lb. If $1\frac{1}{2}$% of the weight was lost in the finishing process, what was the original weight of the casting?

13. What is the yearly tax on a piece of property that is assessed for $5,700 if the tax rate is 0.03119, or $31.19 per $1,000?

25. Interest. Interest is a special application of percentage and by definition applies only to money. *It is defined as money paid for the use of money.* The amount upon which the interest is computed is called the principal. In considering simple *percentage problems*, time is not involved, but in interest problems, time must be considered. For instance, in Example 2–25 we found $37\frac{1}{2}$% of 240 and said nothing about time. Now, in a similar

interest problem, we might find 6% of \$240 but it must be specified for how long a period the interest is to run. If the time is 1 year, 6% of \$240 is $0.06 \times \$240$ or \$14.40, but if the time is 2 years, 6% of \$240 for 2 years is $0.06 \times \$240 \times 2$, which gives \$28.80 or 2 times the first answer. A time period of 3 years would give $3 \times \$14.40$ or \$43.20. A period of 6 months would give ½ of \$14.40 or \$7.20. These are examples of simple interest.

If a man should borrow \$1,000 for 1 year and agree to pay 6% interest, he would have to pay back the principal of \$1,000 plus 6% of \$1,000, which is the interest. Thus, the amount paid back is \$1,000 + 60 = \$1,060. This is called the *amount*. The \$1,000 is the *principal* and the \$60 is the *interest*.

However, when borrowing money from a bank it is customary for the bank to deduct the total interest at the time the loan is made. Therefore, in the case cited, the borrower will receive only \$940, since the interest of \$60 is deducted at once. But he will have to pay back \$1,000, although he has had only \$940 to use. This, of course, amounts to considerably more than 6% interest.

On a savings bank account, the interest is computed each six months and then added to the principal to make a new principal for the following six months. This is called *compound interest.*

Interest may be compounded each year or annually, each six months or semiannually, each three months or quarterly, or even every month, which would be called monthly. In the case of the savings bank account, the interest is compounded semiannually.

Compound interest differs from simple interest in that the interest at the end of the compounding period becomes part of the principal, whereas in simple interest the principal is not increased in value by the addition of the interest.

EXAMPLE 2–32. Interest is compounded annually at 6% on \$100. What is the amount at the end of 2 years?

Solution:

$$
\begin{aligned}
6\% &= 0.06. \\
0.06 \times \$100 \times 1 &= \$6.00 \text{ interest for first year,} \\
\$100 + \$6.00 &= \$106 \text{ amount at end of first year,} \\
\$106 \times 0.06 &= \$6.36 \text{ interest for second year,} \\
\$106 + \$6.36 &= \$112.36 \text{ amount at end of second year.}
\end{aligned}
$$

EXAMPLE 2–33. Interest is compounded semiannually at 6% on \$100. Find the total amount at the end of 2 years.

Solution: Since the interest is compounded semiannually, it must be added to the principal at the end of 6 months and this amount taken as the new principal for the next 6 months. Thus,

$$
\begin{aligned}
0.06 \times \$100 \times \tfrac{1}{2} &= \$3.00 \text{ interest for 6 months,} \\
\$100 + \$3.00 &= \$103.00 \text{ amount at end of 6 months,}
\end{aligned}
$$

$$0.06 \times \$103 \times \tfrac{1}{2} = \$3.09 \text{ interest for second 6 months,}$$
$$\$103 + \$3.09 = \$106.09 \text{ amount at end of 1 year,}$$
$$0.06 \times \$106.09 \times \tfrac{1}{2} = \$3.18 \text{ interest for third 6 months,}$$
$$\$106.09 + \$3.18 = \$109.27 \text{ amount at end of 18 months,}$$
$$0.06 \times \$109.27 \times \tfrac{1}{2} = \$3.28 \text{ interest for fourth 6 months,}$$
$$\$109.27 + \$3.28 = \$112.55 \text{ total amount at end of 2 years.}$$

26. Discount. All wholesale dealers employ what is known as a discount rate in selling their products to distributors and retailers.

A discount is a percentage allowed by the wholesale dealer to the retailer, this percentage being deducted from the original price and the remainder being what the retailer must pay. The original price is quite generally put into the catalog of the wholesale manufacturer as a list price. For example, a transformer may have a list price of \$500 but the manufacturer allows a discount rate of 25%. This means that 25% of \$500 may be deducted, leaving the net cost of the transformer as \$500 less \$125, or \$375.

Sometimes such discount rates are put in as 10–10–5 or 15–5, or some similar grouping. A 10–10–5 discount means that 10% is first taken on the list price and then deducted. Then 10% is taken of what remains and deducted. Finally, 5% is taken of this last remainder and deducted.

EXAMPLE 2–34. What is the true cost of an article listed at \$100 with a discount rate of 20% allowed?

Solution:
$$20\% \text{ of } \$100 = \$20,$$
$$\$100 - \$20 = \$80 \text{ true cost.}$$

EXAMPLE 2–35. What is the actual price of a transformer listed at \$800.00 with a discount rate of 10–10–5 allowed?

Solution:
$$10\% \text{ of } \$800.00 = \$80.00,$$
$$\$800.00 - \$80.00 = \$720.00,$$
$$10\% \text{ of } \$720.00 = \$72.00,$$
$$\$720.00 - \$72.00 = \$648.00,$$
$$5\% \text{ of } \$648.00 = \$32.40,$$
$$\$648.00 - \$32.40 = \$615.60 \text{ actual price.}$$

It should be noted that this discount rate of 10–10–5 does not represent a discount of 25%, as might readily be supposed. 25% of \$800.00 would be \$200.00, and this deducted from the \$800.00 would leave \$600.00 as the actual price, whereas it was found that the final actual price at a discount of 10–10–5 is \$615.60. So it can be readily seen that this discount rate of 10–10–5 represents a discount of approximately 23% instead of 25%.

EXERCISE 2–5

1. Find what \$1,725 will amount to in 5 yr at $4\frac{1}{2}$% simple interest.
2. Find what \$1,000 will amount to in 3 yr at 5% interest, if compounded semiannually.

3. Find what $1,200 will amount to in 2 yr at 2½% interest, if compounded quarterly.

4. A man wishes to borrow $2,500 for a period of 3 months. If the legal rate of interest is 5%, how much will he pay back to clear up the loan?

5. The list price on a piece of equipment is $435. The discount rate is 15–10–5. What is the selling price of the piece of equipment?

6. With a discount rate of 15–10, what is the actual cost of an article listed at $1,500?

27. Ratio. The relationship of one quantity to another may be shown in several ways, but one convenient way to express the relationship is by means of a ratio. *A ratio is the relation of two quantities that is expressed as the quotient of the first divided by the second, either implied or actually divided.*

Thus, the ratio of $6 to $3 can be written as $6/$3, or as 2, the former being an implied division, and the latter being the result of the performed division. It can be written also as $6 : $3, read $6 is to $3.

The two quantities used in a ratio are called the *terms.* The first term is the *antecedent* and the second term is the *consequent.* Both terms form a *couplet.*

A ratio, being an indicated division, is a fraction, and the rules that pertain to fractions also apply to the ratio. The antecedent is the dividend or numerator of the fraction and the consequent is the divisor or denominator of the fraction.

A ratio can be stated between two quantities only when those quantities are expressed in the same units. Thus, a ratio can be expressed between $6 and $3 but not between $6 and 3 pounds. *The result or quotient of the ratio is always an abstract number,* even though the terms are expressed in some unit.

An inverse ratio is the ratio of the reciprocals of two quantities, and may be expressed by interchanging the terms. Thus,

$$\text{inverse ratio of } \$6 : \$3 = \frac{1}{\$6} : \frac{1}{\$3},$$

but since

$$\frac{1}{\$6} \div \frac{1}{\$3} = \frac{\frac{1}{\$6}}{\frac{1}{\$3}} = \frac{\$3}{\$6},$$

the inverse ratio can be written as $3 : $6.

Example 2–36. In order to determine the speed of a motor, it was necessary to attach a small disk with a rubber rim to the tachometer and hold this disk against the rim of a pulley on the motor shaft. The speeds of the motor and tachometer then would be in an inverse ratio to the diameters. The diameter of the disk on the tachometer was 1 in. and the

diameter of the pulley on the motor was 4 in. If the tachometer read 3,600 rpm, what was the speed of the motor?

Solution:
$$\frac{\text{speed of motor}}{\text{speed of tachometer}} = \frac{1}{4}.$$

The speed of the motor is ¼ the speed of the tachometer because the pulley diameter is 4 times as large as the disk diameter. Therefore, speed of motor = ¼(3,600) = 900 rpm.

28. Proportion. *A statement of equality between two ratios is called a proportion.* \$6 : \$3 and \$10 : \$5 are equal ratios and therefore can be formed into the proportion \$6 : \$3 = \$10 : \$5. It is not essential that each ratio refer to the same quantities. Thus, \$6 : \$3 = 4 men : 2 men is also a true proportion.

The first and last terms of a proportion are called the *extremes;* the second and third terms are called the *means.* In any proportion, the following relationships will hold true:

1. *The product of the extremes is equal to the product of the means.*
2. *The product of the extremes divided by one mean gives the other mean.*
3. *The product of the means divided by one extreme gives the other extreme.*

Two ratios may form a direct proportion or they may form an inverse proportion. *If both ratios increase or decrease together, then they form a direct proportion, but if one decreases while the other increases, then they form an inverse proportion.* In the first case, one direct ratio is equal to a second direct ratio, but in the second case, one direct ratio is equal to a second inverse ratio. For example, the resistance of a copper conductor increases as its length is increased. Therefore, a direct proportion can be set up between the ratios of the resistances and lengths of two conductors. The proportion would be written

$$\frac{R_1}{R_2} = \frac{l_1}{l_2}$$

where R_1 and l_1 are the resistance and length of the first conductor and R_2 and l_2 are the resistance and length of the second conductor. The two ratios form a direct proportion because the resistance of the first conductor is in the same ratio to the resistance of the second as the length of the first is to the length of the second.

Again, the resistance of a copper conductor decreases as its area is increased. Therefore, an inverse proportion can be set up between the ratios of the resistances and areas of two conductors. The proportion would be written

$$\frac{R_1}{R_2} = \frac{A_2}{A_1}$$

where R_1 and A_1 are the resistance and area of the first conductor and R_2 and A_2 are the resistance and area of the second conductor. Attention is

called to the fact that this proportion is inverted compared with the previous one.

The solution of a proportion is used extensively in problems involving the lever. A stiff bar or rod supported at some point and supporting a weight at some other point is called a *lever*. The point of support for the bar or rod is called a *fulcrum*. Fig. 2–1 shows various types of levers. In each example in the figure the forces P and W are in an inverse ratio to the distance from the fulcrum or

$$\frac{P}{W} = \frac{l_2}{l_1}.$$

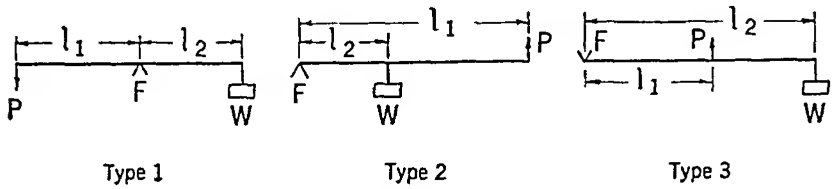

Fig. 2-1.

Thus, it may be seen that a small force P may be used to support a large weight W provided it is placed farther from the fulcrum than the weight.

EXAMPLE 2–37. In Type 3 what force placed 3 ft from the fulcrum is required to support a weight of 100 lb placed 5 ft from the fulcrum?

Solution:

$$\begin{aligned} P : W &= l_2 : l_1, \\ P : 100 &= 5 : 3, \\ 3P &= (100)(5), \\ P &= \frac{(100)(5)}{3} = \frac{500}{3} = 166\frac{2}{3} \text{ lb.} \end{aligned}$$

EXERCISE 2–6

1. It takes 22 men 18 days to do a certain piece of work. How long will it take 12 men to do the same amount of work if they work at the same rate?

2. The resistance of a copper conductor 150 ft long is 15.75 ohms. What is the resistance of a conductor of the same size that is 85 ft long?

3. In the Type 2 lever, what force will be required to lift a 250-lb weight located 8 in. from the fulcrum, if the force is applied $2\frac{1}{2}$ ft from the fulcrum?

4. In 25 days of 6 hr each, 8 men can unload 1,250 cu yd of gravel. How many men will be necessary to do the same work in 10 days working 8 hr per day?

29. Powers. When several numbers are multiplied together each number is called a *factor* and the result of the multiplication is called the *product*. Thus, $3 \times 4 \times 2 \times 5 = 120$. The numbers 3, 4, 2, and 5 are factors and 120 is the product. If all the factors are alike then the product is called the *nth power* of that factor, where n represents the number of

factors. Thus, $2 \times 2 \times 2 \times 2 = 16$, in which 16 is a power of the factor 2. Therefore, when a number is used two times as a factor, it is said to be squared or raised to the second power; if it is used three times as a factor, it is said to be cubed or raised to the third power; four times, the fourth power; five times, the fifth power, and so on.

Instead of writing $2 \times 2 \times 2 \times 2 = 16$, we may write the expression $2^4 = 16$. The small number 4 written a little above and to the right of 2 indicates how many times 2 is taken as a factor and is called an *exponent*. Thus, $5^3 = 125$ means that 5 is taken 3 times as a factor and the product or power of 5 obtained is 125. This gives a short method of writing the continued product of the same number. Illustrations:

$$
\begin{aligned}
2 \times 2 \times 2 &= 2^3 = 8, \\
3 \times 3 \times 3 \times 3 \times 3 &= 3^5 = 243, \\
5 \times 5 \times 5 \times 5 &= 5^4 = 625.
\end{aligned}
$$

30. Roots. *A root of a number is one of the equal factors that when multiplied together give the number.* The process of finding a root is therefore the inverse of the process of finding a power. If 9 is divided into its two equal factors, 3×3, one of the factors is a root of 9; if 125 is divided into its three equal factors, $5 \times 5 \times 5$, one of these factors is a root of 125. The square root of a number is one of its two equal factors; the cube root, one of its three equal factors; the fourth root, one of its four equal factors, and so on.

A root of a number is indicated by the sign $\sqrt[n]{\ }$, called a radical sign, where n denotes the root to be taken. If n is omitted, only the second, or square, root is understood. Illustrations:

$$
\begin{aligned}
\sqrt{64} &= \text{square root of } 64 = 8, \\
\sqrt[3]{27} &= \text{cube root of } 27 = 3, \\
\sqrt[4]{16} &= \text{fourth root of } 16 = 2.
\end{aligned}
$$

Frequently it is necessary to extract the square root of a number by arithmetic means. A method for doing this follows. Cube root is ordinarily found by means of the slide rule or logarithms, and higher roots, when required, are found by means of logarithms.

EXAMPLE 2–38. Find $\sqrt{416{,}025}$.

Solution:

$$
\begin{array}{r|l}
 & 6\ \ 4\ \ 5 \\
 & \sqrt{41'60'25} \\
 & 36 \\
\text{trial divisor} = 2 \times 60 = 120 & 5\,60 \\
4 & 4\,96 \\
\text{complete divisor} = 120 + 4 = 124 & \\
\text{trial divisor} = 2 \times 640 = 1{,}280 & 64\,25 \\
5 & \\
\text{complete divisor} = 1{,}280 + 5 = 1{,}285 & 64\,25
\end{array}
$$

Starting at the decimal point, separate the number into groups with two digits in a group as shown by the marks placed between the 2 and 0 and between the 6 and 1. The number is now separated into three groups, 41′60′25, and this means there will be three digits in the answer, one digit for each group of two.

Find the greatest square, 36, that is contained in the left-hand group and write its root, 6, directly above the 41 as the first figure of the required root. Write 36, the square of 6, under the left-hand group, 41, and subtract, giving a remainder of 5. Then bring down the next group, 60, at the right of 5, making a new dividend of 560.

As a trial divisor of 560, double the 6, the root already found, and annex a zero, giving $2 \times 60 = 120$. Divide 560 by this trial divisor, 120, to obtain the next figure of the root. This is found to be 4. Write this 4 as the second figure of the required root, directly above the second group of digits. Also add the 4 to the trial divisor, making a complete divisor of $120 + 4$, or 124. Then multiply the complete divisor, 124, by the second root, 4, and write down the result, 496, under the dividend 560. Subtract 496 from 560 for a remainder of 64, and bring down the next group of digits, 25, making a new dividend of 6,425.

Now, as a trial divisor, double the root, 64, and annex a zero, giving $2 \times 640 = 1{,}280$. Divide the dividend 6,425 by this trial divisor of 1,280 to obtain the next figure in the root. This will be 5. Write the 5 as the next figure of the root, directly above the 25, the last group of digits. Also, add the 5 to the trial divisor, making a complete divisor of 1,285. Multiply the complete divisor, 1,285, by 5, the root just found, and put the result, 6,425, under the dividend 6,425. Since 6,425 subtracted from 6,425 leaves zero and there are no more groups of digits to bring down, the process is completed and the square root of 416,025 is found to be exactly 645 with no remainder.

EXAMPLE 2–39. Find $\sqrt{2{,}925.0782}$.

Solution:

		5 4. 0 8 3
		$\sqrt{29'25.07'82'00}$
trial divisor = 2×50 =	100	25
	4	4 25
complete divisor = $100 + 4$ =	104	4 16
1st trial divisor = 2×540 =	1,080	9 07 82
	0	
1st complete divisor =	1,080	
2nd trial divisor = $2 \times 5{,}400$ =	10,800	
	8	8 64 64
2nd complete divisor = $10{,}800 + 8$ =	10,808	43 18 00
trial divisor = $2 \times 54{,}080$ =	108,160	
	3	32 44 89
complete divisor = $108{,}160 + 3$ =	108,163	10 73 11 remainder

This square root is found in the same manner as in Example 2–38. Starting at the decimal point, the number is separated into groups of two digits each in both directions. This separates the number into four groups with two groups at the left of the decimal point and two groups at the right of the decimal point. Therefore, the answer will have two places at the left of its decimal point, since it will have one place for each group of two digits in the original number.

The largest perfect square in the first group is 25 and its root is 5. The root, 5, is placed directly above the first group and the square, 25, is placed under 29 and subtracted therefrom, leaving a remainder of 4. The next group, 25, is brought down beside the 4, giving a new dividend of 425.

As a trial divisor, the root, 5, is doubled and a zero annexed, giving $2 \times 50 = 100$. Dividing 100 into the dividend, 425, gives the second root of 4. This root, 4, is put in its proper place above the second group of digits, 25, and is also added to the trial divisor, 100, making a complete divisor of 104. Now the second root, 4, is multiplied by the complete divisor, 104, giving a total of 416, which is subtracted from 425 to give a remainder of 9. Then the next group, 07, is brought down beside the remainder, 9, making a dividend of 907.

As a trial divisor, the root 54 is doubled and a zero annexed, giving a trial divisor of 1,080. Since 1,080 will not divide into 907, a zero is placed in the answer for the next root, and the next group of digits, 82, brought down beside the 907 to give a new dividend of 90,782. Also, a second trial divisor is formed by doubling 540 and annexing a zero, thereby giving $2 \times 5{,}400$ or 10,800. This second trial divisor is divided into the new dividend 90,782 to give 8, which is attached to the answer as another root. It is also added to the trial divisor to give a second complete divisor of 10,808.

Multiplying the second complete divisor by the last root, 8, gives 86,464, which is subtracted from 90,782 to give a remainder of 4,318.

If it is desired to carry the answer out to three decimal places, two zeros are added to the original number to give another group of digits. This group of digits is brought down beside the remainder 4,318 to give a new dividend of 431,800. The root, 5,408, is doubled and a zero annexed to give a trial divisor of 108,160. When divided into the dividend of 431,800, it is found to be contained not quite 4 times. Therefore, the next figure in the root will have to be 3 and when this 3 is added to the trial divisor, the complete divisor becomes 108,163. Multiplying the complete divisor by the last root, 3, gives 324,489, which is subtracted from 431,800 to leave a remainder of 107,311. Since this remainder makes more than one half a unit when used in fraction form with the divisor thus, $\frac{107{,}311}{108{,}160}$, the last figure of the root is increased to 4. Therefore, the square root of 2,925.0782 to three decimal places is 54.084.

31. Checking square root by the residue of nines. The square root of a number can be checked by the residue of nines. The square of the residue of nines in the root plus the residue of nines in the remainder should equal the residue of nines in the original number. Thus, checking Example 2-39

$$\begin{aligned}
\text{root} &= \quad \cancel{5}\ \cancel{4}.\ 0\ \ 8\ \ 3 \longrightarrow \text{residue} = 11 \longrightarrow 2\\
\text{number} &= \sqrt{2\cancel{9}'2\cancel{5}.07'8\cancel{2}'00} \longrightarrow \text{residue} = 8\\
\text{remainder} &= \quad 10\ 73\ \cancel{1}\cancel{1} \longrightarrow \text{residue} = 4
\end{aligned}$$

Therefore,

$$\begin{array}{ccccc}
(\text{residue of root})^2 & + & \text{residue of remainder} & = & \text{residue of number}\\
(2)^2 & + & 4 & = & 8\\
4 & + & 4 & = & 8
\end{array}$$

32. Scientific notation. In engineering and scientific work, it is frequently necessary to use very large or very small numbers. For instance, the number of electrons that must move past every point in an electric circuit per second to produce 1 ampere of current is a number of such a size that it is beyond human conception when written out in full. Its value is 6,300,000,000,000,000,000. Again the accepted value today for the charge on the electron is something like 0.00000000048 electrostatic unit, a number so small that it too is beyond our conception. So, in order to express such numbers in simpler form and make them understandable, powers of 10 are employed. Thus, 6,300,000,000,000,000,000 $= 6.3 \times$ 1,000,000,000,000,000,000. But 1,000,000,000,000,000,000, $= 10^{18}$ because 10 is used 18 times as a factor. Therefore, 6,300,000,000,000,000,000 $= 6.3 \times 10^{18}$. Also

$$0.00000000048 = 4.8 \times 0.0000000001.$$

But $$0.0000000001 = \frac{1}{10{,}000{,}000{,}000} = \frac{1}{10^{10}} = 10^{-10}.$$

So $$0.00000000048 = 4.8 \times 10^{-10}.$$

Therefore, to express in scientific notation any number given in the common form, factor out the proper power of 10 to place the decimal point where desired. *Usually the decimal point is located after the first significant figure.* Thus, $28{,}150{,}000 = 2.815 \times 10^7$. This form is preferred to 28.15×10^6 or 281.5×10^5, although the latter two are equal to the former.

To write in common form any number that is expressed in scientific notation, simply perform the indicated multiplication, or, in other words, move the decimal point the number of places indicated by the exponent and attach as many zeros as may be needed. *If the exponent is positive, the decimal point is moved to the right, whereas if the exponent is negative, the decimal point is moved to the left.* Thus,

$3.14 \times 10^6 = 3{,}140{,}000$ (point moved 6 places to the right).
$5.24 \times 10^{-5} = 0.0000524$ (point moved 5 places to the left).

Numbers expressed in scientific notation can be added or subtracted only if like powers of 10 are used. The numbers are added or subtracted as required and the same power of 10 is kept with the answer. If the numbers do not have the same power of 10, they must be converted to the same power before addition or subtraction can be applied.

EXAMPLE 2–40. Add and subtract the following: 2.852×10^6 and 1.987×10^6.

Solution:

Addition

$$\begin{array}{r} 2.852 \times 10^6 \\ 1.987 \times 10^6 \\ \hline 4.839 \times 10^6 \end{array} \text{ Ans.}$$

Subtraction

$$\begin{array}{r} 2.852 \times 10^6 \\ 1.987 \times 10^6 \\ \hline 0.865 \times 10^6 \end{array} \text{ Ans.}$$

EXAMPLE 2–41. Add and subtract the following: 4.58100×10^8 and 2.731×10^6.

Solution: One of the numbers must be converted into the same power of 10 as the other before addition or subtraction may be used. Thus, $4.58100 \times 10^8 = 458.100 \times 10^6$. Now we can add or subtract.

Addition

$$\begin{array}{r} 458.100 \times 10^6 \\ 2.731 \times 10^6 \\ \hline 460.831 \times 10^6 \end{array} \text{ Ans.}$$

Subtraction

$$\begin{array}{r} 458.100 \times 10^6 \\ 2.731 \times 10^6 \\ \hline 455.369 \times 10^6 \end{array} \text{ Ans.}$$

If it is desired to multiply numbers expressed in scientific notation, the numbers are multiplied and the powers of 10 are added. If it is desired to divide one number by another when both are expressed in scientific notation, the division of the numbers is performed and the powers of 10 are subtracted. Thus, 3.14×10^6 multiplied by 2.50×10^4 gives 7.85×10^{10} and 6.28×10^6 divided by 3.14×10^2 gives 2.00×10^4.

EXAMPLE 2–42. Solve the following and write the answer in scientific notation:

$$\text{(a)} \quad \frac{(25{,}400{,}000)(3{,}750{,}000{,}000)}{(22{,}900)(1{,}520{,}000)}.$$

Solution:

$$\begin{aligned} \frac{(25{,}400{,}000)(3{,}750{,}000{,}000)}{(22{,}900)(1{,}520{,}000)} &= \frac{(2.54 \times 10^7)(3.75 \times 10^9)}{(2.29 \times 10^4)(1.52 \times 10^6)} \\ &= \frac{(2.54)(3.75)(10^{16})}{(2.29)(1.52)(10^{10})} \\ &= 2.73 \times 10^6. \end{aligned}$$

$$\text{(b)} \quad \frac{(1{,}450)(32{,}500)}{75{,}500{,}000{,}000}.$$

Solution: $$\frac{(1{,}450)(32{,}500)}{75{,}500{,}000{,}000} = \frac{(1.45 \times 10^3)(3.25 \times 10^4)}{7.55 \times 10^{10}}$$
$$= \frac{(1.45)(3.25)(10^7)}{7.55 \times 10^{10}}$$
$$= 0.625 \times 10^{-3} = 6.25 \times 10^{-4}.$$

33. *Applications of square roots.* In geometry there is the theorem of Pythagoras which states that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs. This theorem provides a most useful application of square root.

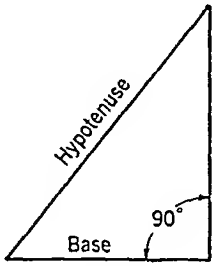

Fig. 2–2.

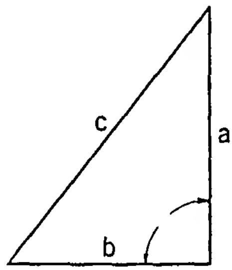

Fig. 2–3.

A right triangle is a triangle in which one of the angles is equal to 90°, or a right angle. The side opposite the right angle is called the hypotenuse and the other two sides are called the legs. One of the legs will be the base and the other will be the altitude. (See Fig. 2–3.) If the hypotenuse is denoted by c, the altitude by a and the base by b (see Fig. 2–3), then, by the theorem of Pythagoras.

$$c^2 = a^2 + b^2.$$

Extracting the square root of both sides: $c = \sqrt{a^2 + b^2}$. If the hypotenuse and altitude are given, then to find the base, b

$$b^2 = c^2 - a^2,$$

or

$$b = \sqrt{c^2 - a^2}.$$

Likewise if the hypotenuse and base are given, then to find the altitude, a,

$$a^2 = c^2 - b^2,$$

or

$$a = \sqrt{c^2 - b^2}.$$

Example 2–43. The two legs of a right triangle are $a = 12.5$ in. and $b = 15.25$ in. What is the hypotenuse?

Solution:
$$c^2 = a^2 + b^2,$$
$$c = \sqrt{a^2 + b^2},$$
$$c = \sqrt{(12.5)^2 + (15.25)^2},$$
$$c = \sqrt{388.8125} = 19.718 \text{ in.}$$

Example 2–44. The hypotenuse of a right triangle is 14.3 in. and the base is 9.52 in. What is the altitude?

Solution:

$$a = \sqrt{c^2 - b^2},$$
$$a = \sqrt{(14.3)^2 - (9.52)^2},$$
$$a = \sqrt{113.8596} = 10.67 \text{ in.}$$

EXERCISE 2-7

Find the square roots of the following and check each answer by the residue of nines:

1. 309.2545. **2.** 0.062875. **3.** 56.4001. **4.** 0.64281.

5. Find the 4th root of 0.00001296.
(*Hint:* Find the square root two times.)

6. Find the 4th root of 0.14776336.

7. Write the following in scientific notation:

(a) 0.000000000000000000159 coulomb (the charge on an electron).
0.0000000000002 cm (the radius of the hydrogen electron).
64,310,000,000,000.
0.0000000000000000000000000009 gram (mass of the electron).

(b) Change the following from scientific notation form to the common form:

1.43×10^{12}. 6.71×10^{9}.
4.82×10^{-10}. 8.39×10^{-15}.

Solve the following and write your answer in scientific notation form:

8. $\dfrac{(382,000)(2,450,000)}{(9,250,000,000)(53,000)}$. **9.** $\dfrac{(74,300,000)(876,000,000)}{(3,750,000)(485,000)}$.

10. The base of a right triangle is 1.5 ft and the altitude is 1.3 ft. Find the hypotenuse.

11. The hypotenuse of a right triangle is 4 ft $9\frac{1}{2}$ in. and the base is 2 ft $8\frac{1}{4}$ in. Find the altitude.

12. A room is 9 ft 6 in. wide and 14 ft 8 in. long. Find the distance between opposite corners.

13. Find the diagonal of a $2\frac{1}{2}$-in. $\times$ $4\frac{1}{4}$-in. negative.

14. In the expression $V = \sqrt{2gh}$, V is the velocity in feet per second that a body will have falling from height h. Find the value of V for a body that has fallen 250 ft. Use $g = 32$.

15. Two alternating-current voltages at right angles with each other are equal to 114 and 85 respectively. What will be the total voltage (the hypotenuse of the right triangle formed)?

16. A tower 127 ft high has a guy wire anchored 200 ft from its base. What is the length of the guy wire?

34. Weights and measures. There are two systems of weights and measures in common use today, the *English* and the *Metric*. The former is in everyday use throughout the United States, whereas the latter is used

in engineering and scientific work as well as throughout Europe. The English system has no common factor to tie the various units together and therefore is difficult to handle. The metric system, on the other hand, has the common factor 10 running all the way through it and this makes it simpler and much easier to handle than the English system. The common terms in the metric system are:

meter—unit of length	deci—one tenth of
liter—unit of volume	deka—ten times
gram—unit of weight	hecto—one hundred times
milli—one thousandth of	kilo—one thousand times
centi—one hundredth of	

Tables of units for both systems and a table of conversion units follow. (*Note:* A table of abbreviations for units of measure will be found on page 63.)

THE ENGLISH SYSTEM

(1) *Measures of weight:*

7,000 grains	= 1 pound
16 ounces	= 1 pound
100 pounds	= 1 hundredweight
2,000 pounds	= 1 ton
2,240 pounds	= 1 long ton
62½ pounds	= 1 cubic foot of water

(2) *Measures of length:*

12 inches	= 1 foot
3 feet	= 1 yard
5½ yards or 16½ feet	= 1 rod
320 rods	= 1 mile
5,280 feet	= 1 mile

(3) *Measures of area:*

144 square inches	= 1 square foot
9 square feet	= 1 square yard
30¼ square yards	= 1 square rod
160 square rods	= 1 acre
640 acres	= 1 square mile

(4) *Measures of volume:*

1,728 cubic inches	= 1 cubic foot
27 cubic feet	= 1 cubic yard
128 cubic feet	= 1 cord

(5) *Liquid measure:*

4 gills	= 1 pint
2 pints	= 1 quart
4 quarts	= 1 gallon
31½ gallons	= 1 barrel
231 cubic inches	= 1 gallon

(6) *Dry measure:*

2 pints	= 1 quart
8 quarts	= 1 peck
4 pecks	= 1 bushel
2,150.42 cubic inches	= 1 bushel

(7) *Time measure:*

60 seconds	=	1 minute
60 minutes	=	1 hour
24 hours	=	1 day
365 days	=	1 common year
366 days	=	1 leap year

The Metric System

(1) *Measures of weight, or force:*

10 milligrams	=	1 centigram
10 centigrams	=	1 decigram
10 decigrams	=	1 gram
10 grams	=	1 dekagram
10 dekagrams	=	1 hectogram
10 hectograms	=	1 kilogram

(2) *Measures of length:*

10 millimeters	=	1 centimeter
10 centimeters	=	1 decimeter
10 decimeters	=	1 meter
10 meters	=	1 dekameter
10 dekameters	=	1 hectometer
10 hectometers	=	1 kilometer

(3) *Measures of area:*

100 square millimeters	=	1 square centimeter
100 square centimeters	=	1 square decimeter
100 square decimeters	=	1 square meter
100 square meters	=	1 square dekameter
100 square dekameters	=	1 square hectometer
100 square hectometers	=	1 square kilometer

(4) *Measures of volume:*

1,000 cubic millimeters	=	1 cubic centimeter
1,000 cubic centimeters	=	1 cubic decimeter = 1 liter
1,000 cubic decimeters	=	1 cubic meter = 1 kiloliter

(5) *Measures of capacity:*

10 milliliters	=	1 centiliter
10 centiliters	=	1 deciliter
10 deciliters	=	1 liter = 1 cubic decimeter
10 liters	=	1 dekaliter
10 dekaliters	=	1 hectoliter
10 hectoliters	=	1 kiloliter

Conversion Tables

(1) *Measures of length:*

1 inch	=	2.54 centimeters
1 foot	=	30.48 centimeters
1 meter	=	39.37 inches
1 mile	=	1.609 kilometers

(2) *Measures of area:*

1 circular mil	=	0.7854 square mils
1 circular mil	=	0.000507 square millimeters
1 square inch	=	6.452 square centimeters
1 square meter	=	10.76 square feet

(3) *Measures of volume:*

1 cubic inch	=	16.39 cubic centimeters
1 cubic centimeter	=	0.0610234 cubic inches
1 cubic foot	=	28.317 liters
1 pint (dry)	=	550.614 cubic centimeters
1 pint (liquid)	=	473.179 cubic centimeters
1 liter	=	0.2642 gallon

(4) *Measures of weight, or force:*

1 gram	=	981 dynes
1 ounce	=	28.35 grams
1 kilogram	=	2.205 pounds

(5) *Measures of temperature:*

1 degree Fahrenheit	=	$\frac{5}{9}$ degree centigrade
−40 degrees Fahrenheit	=	−40 degrees centigrade
0 degrees Kelvin or Absolute	=	−459.4 degrees Fahrenheit or −273.13 degrees centigrade

Abbreviations for Units of Measure

(Abbreviations are the same for singular and plural.)

Units of the foregoing tables not given here are standardly spelled out.

barrel	bbl
bushel	bu
centigram	cg
centiliter	cl
centimeter	cm
circular mil	cir mil
cord	cd
cubic centimeter	cc or cm^3
cubic decimeter	cu dm or dm^3
cubic foot	cu ft
cubic inch	cu in.
cubic meter	cu m or m^3
cubic millimeter	cu mm or mm^3
cubic yard	cu yd
decigram	dg
deciliter	dl
decimeter	dm
dekagram	Dg
dekaliter	Dl
dekameter	Dm
foot	ft
gallon	gal
grain	gr
gram	g
hectogram	Hg
hectoliter	Hl
hectometer	Hm
hour	hr
hundredweight	cwt
inch	in.
kilogram	kg
kiloliter	kl
kilometer	km
meter	m
milligram	mg
milliliter	ml
millimeter	mm
minute	min
ounce	oz
peck	pk
pint	pt
pound	lb
quart	qt
second	sec
square centimeter	sq cm or cm^2
square decimeter	dm^2
square dekameter	Dm^2
square foot	sq ft
square hectometer	Hm^2
square inch	sq in.
square kilometer	sq km or km^2
square meter	sq m or m^2
square mil	sq mil or mil^2
square mile	sq mile
square millimeter	sq mm or mm^2
square rod	sq rod
square yard	sq yd
yard	yd
year	yr

Frequently it is desirable to change from one unit to another within a system or from a unit in one system to a corresponding unit in the other system. For instance, it is often necessary to change the speed of an automobile from miles per hour to feet per second, or measurements made in the metric system must often be changed into similar terms in the English system. There are countless cases where these kinds of changes are necessary.

EXAMPLE 2–45. Reduce 25 yd 3 ft 9 in. to inches.

Solution:

$$\begin{aligned} 25\text{ yd} &= 25 \times 3\text{ ft} = 75\text{ ft} = 75 \times 12\text{ in.} = 900\text{ in.} \\ 3\text{ ft} &= 3 \times 12\text{ in.} = 36\text{ in.} \\ 9\text{ in.} &= 9\text{ in.} \\ &\overline{945\text{ in.}} \end{aligned}$$

EXAMPLE 2–46. Find the weight of a gallon of water.

Solution: 1 cu ft of water weighs 62.4 lb. Since there are 231 cu in. in 1 gal there will be $\frac{231}{1,728}$ cu ft in 1 gal. So the weight of 1 gal of water is

$$\frac{231}{1,728} \times 62.4 = 8.342\text{ lb} = 8\text{ lb }5.47\text{ oz.}$$

EXAMPLE 2–47. Change 35 mph to feet per second.

Solution:

$$5,280\text{ ft} = 1\text{ mile},$$
$$3,600\text{ sec} = 1\text{ hr.}$$

So

$$\text{feet per second} = (35)\,\frac{\text{feet in 1 mile}}{\text{seconds in 1 hr}},$$
$$\text{feet per second} = (35)\,\frac{5,280}{3,600} = 51.4.$$

EXAMPLE 2–48. A world record for skating was established when a skater made 1,500 m in 2 min 20 sec. What time would this mean for a mile?

Solution:

$$\begin{aligned} 1\text{ mile} &= 1.609\text{ km} = 1,609\text{ m}, \\ 2\text{ min }20\text{ sec} &= (2 \times 60 + 20)\text{ sec} = 140\text{ sec}, \\ \text{seconds per mile} &= (\text{seconds per meter})(\text{meters per mile}), \end{aligned}$$

or

$$\frac{\text{seconds}}{\text{mile}} = \left(\frac{\text{seconds}}{\text{meter}}\right)\left(\frac{\text{meters}}{\text{mile}}\right),$$

$$\text{seconds per mile} = \frac{140}{1,500}\,(1,609) = 150.2\text{ sec},$$

$$150.2\text{ sec} = \frac{150.2}{60}\text{ min} = 2\text{ min }30.2\text{ sec.}$$

To convert temperature expressed in Fahrenheit reading to equivalent temperature expressed in centigrade reading, add 40 to the number of

degrees expressed in Fahrenheit, take 5/9 of this amount and subtract 40 from the product to determine the number of degrees centigrade.

EXAMPLE 2–49. 212° F is equal to how many degrees C?

$$212 + 40 = 252,$$
$$\frac{5}{9} \times 252 = 140,$$
$$140 - 40 = 100° \text{ C}.$$

To convert centigrade to Fahrenheit readings of temperatures, add 40 to the number of degrees expressed in centigrade, take 9/5 of this amount, and subtract 40 from the product to determine the number of degrees Fahrenheit.

EXAMPLE 2–50. 10° C is equal to how many degrees F?

$$10 + 40 = 50,$$
$$\frac{9}{5} \times 50 = 90,$$
$$90 - 40 = 50° \text{ F}.$$

35. Density and specific gravity. In a discussion of the weights of various substances, the terms density and specific gravity are often involved. Since *density is defined as the mass of a body per unit volume* and usually is measured in pounds per cubic foot or in grams per cubic centimeter, the numerical values of the units of weight and mass are assumed to be the same. *Specific gravity is the ratio of the densities of two bodies and is, therefore, an abstract quantity.*

The specific gravities of solids and liquids are based upon water as a standard, whereas the specific gravities of gases are stated in terms of air or hydrogen as a standard.

To determine the specific gravity of a solid or liquid, a ratio is formed between the weight of a cubic foot of the solid and the weight of a cubic foot of water, which is approximately 62.4 lb. If *sp. gr.* represents the specific gravity of the solid and W its weight per cubic foot, then from our definitions

$$sp.\ gr. = \frac{W}{62.4}.$$

This expression can be made a general one by representing the unit volume of the standard by S. Thus, we get $sp.\ gr. = \frac{W}{S}$.

EXAMPLE 2–51. Slate weighs 175 lb per cubic foot. What is its specific gravity?

Solution: $$sp.\ gr. = \frac{W}{62.4} = \frac{175}{62.4} = 2.80.$$

Frequently it is desirable to change from one unit to another within a system or from a unit in one system to a corresponding unit in the other system. For instance, it is often necessary to change the speed of an automobile from miles per hour to feet per second, or measurements made in the metric system must often be changed into similar terms in the English system. There are countless cases where these kinds of changes are necessary.

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This expression can be made a general one by representing the unit volume of the standard by S. Thus, we get $sp.\ gr. = \dfrac{W}{S}$.

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Solution: $$sp.\ gr. = \frac{W}{62.4} = \frac{175}{62.4} = 2.80.$$

EXERCISE 2-8

1. Normal atmospheric pressure is approximately 14.7 lb per square inch. What does this represent in grams per square centimeter?

2. A mile in 3 min 24 sec represents what time for 1,500 m?

3. Find the number of miles in 25 km.

4. Find the number of cubic meters in a cubic yard.

5. A car is traveling 75 ft per second. How many miles per hour does this represent? How many kilometers per hour?

6. What is the difference in length between 15 in. and 38 cm?

7. Find the number of square miles in a square kilometer.

8. Change 12,000 dynes to pounds.

9. The weight of a certain kind of steel is 490 lb per cubic foot. How many grams per cubic centimeter is this?

10. What is the weight in pounds of a pint of water?

11. The area of a rectangle is the product of its length and width. Find the areas of the following rectangles in square inches and in square feet:

(a) 3 ft 4 in. × 5 ft 8 in.
(b) $4\frac{1}{2}$ in. × $5\frac{5}{8}$ in.
(c) 1 m × 1 m.

12. How many square feet in an acre? How many square yards in an acre?

13. What per cent error would result from using 50 m in place of 55 yd?

14. The speed of light is approximately 186,000 miles per second. What would this be in centimeters per second? Express your answer in scientific notation form.

15. The weight of No. 16 gauge sheet iron is 40 oz per square foot. Express this in kilograms per square meter.

16. What is the specific gravity of a substance that contains 54 cu in. and weighs 6.5 lb?

17. Copper weighs 0.3218 lb per cubic inch. What is its specific gravity?

18. Change the following fahrenheit degrees to centigrade degrees: 200°; 85°; 20°; −25°.

19. Change the following centigrade degrees to fahrenheit degrees: 25°; 75°; 92°; −15°.

REVIEW EXERCISE 2-9

Find the square roots of each of the following:

1. 27.834

2. 0.0007835.

3. 5.29×10^6.

4. 4.835×10^{-5}.

5. 0.001438.

Solve each of the following and express your answers in scientific notation:

6. $\dfrac{(94{,}200)(186{,}000)}{(2750)(63{,}000{,}000)}$.

7. $\dfrac{(57{,}600{,}000)(314{,}850)}{(9{,}983{,}701)(628{,}156{,}300)}$.

8. $\dfrac{(78{,}530{,}000)(347{,}000)}{(25{,}360)(0.68905)}$.

Change the following to common form:

9. 1.43×10^{-12}; 2.78×10^{16}.

10. 5.985×10^{-18}; 6.428×10^{10}.

11. In many states the maximum speed on the highways is 50 miles per hour. What will this be in ft per sec? in meters per sec? in kilometers per hr?

12. Gasoline weighs 43.7 lb per cu ft. What is this in lb per cu in.? in kg per cu meter?

13. The hypotenuse of a right triangle is 15.8 in. and the altitude is 6.43 in. What is the base of the triangle?

14. The base of a right triangle is 5.72 in. and its altitude is 7.86 in. What is the hypotenuse?

15. How far from a 160 lb man (P) must the fulcrum of a type 1 lever (see Section 28) be placed in order to balance a weight (W) of 360 lbs, if the total length of the lever is 12 ft?

16. A 400 lb weight is balanced by a force of 550 lbs, $3\frac{1}{2}$ ft away. If a type 3 lever (see Section 28) is used, how long must the lever be?

17. The resistance of 1000 ft of No. 10 wire is approximately *one ohm* and its cross sectional area is 0.00815 sq in. What will be the resistance of a conductor whose cross sectional area is 0.0328 sq in.?

18. What is the specific gravity of a material that weighs 8.5 lbs for 48 cu in.?

19. If it takes 24 men 15 days to do a piece of work, how many men will be required to do the work in a week of 6 days, assuming they all work at the same rate?

20. Find the distance between the opposite corners of a room that is 15 ft 8 in. long and 10 ft 6 in. wide.

21. A stairway has a rise of 7 in. and a tread of 10 in. The distance between the first and second floors is 12 ft 6 in. What is the length of the stair stringer?

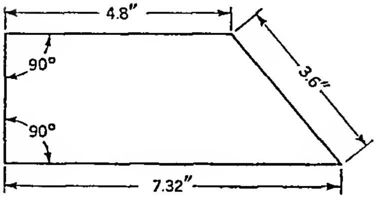
22. White pine weighs 31.2 lb per cu ft. A white pine pattern weighs 1 lb 6 oz. What will be the weight of a brass casting made from the pattern if brass weighs 524 lb per cu ft?

23. Determine the difference in volume between two cubic centimeters and a two-centimeter cube. (*Note:* A cubic inch is the volume of a cube 2.54005 cm on a side whereas an inch cube is 2.54 cm on a side.)

24. Cast iron weighs 449 lb per cu ft. What is the weight of a cast iron ball 8 in. in diameter? (The volume of a sphere is equal to $4.189r^3$ where r is the radius.)

25. Aluminum weighs 165 lb per cu ft. What is its specific gravity? What does this represent in grams per cu cm?

26. Find the number of square inches in the trapezoidal plate shown. The area of a trapezoid is equal to one-half the sum of the parallel sides multiplied by the altitude.



27. No. 2 gage sheet steel has a thickness of 0.266 in. How many sheets can be packed in a box that is 12 in. deep?

28. Ice 6 in. thick is frozen over the surface of a pond having an area of ¼ acre. If the specific gravity of ice is 0.92, what is the weight of the ice in the pond in tons?

29. What is the cost of filling a cylindrical gasoline tank 12 in. in diameter and 4½ ft long if there are 2 gallons in it at the start? Gasoline costs 26.8 cents per gallon. The volume of a cylinder is equal to 0.7854 times the square of the base diameter times the height.

30. If steel weighs 490 pounds per cubic foot, what is the weight of a steel plate ⅜ in. thick, 15 ft long, and 22 in. wide?

31. Determine the weight of a ⅝ in. round steel rod per running foot.

32. Find the weight of a ¾ in. square brass rod per running foot. Brass weighs 524 lb per cu ft.

33. Which is the larger value, 25 in. or 65 cm?

34. How many km are there in 28 miles?

35. An automobile tire has a diameter of 28 in. How many revolutions will the wheel make on a 25 mile trip? The circumference of a circle is π times the diameter. Express your answer in scientific notation.

36. Determine the list price of a piece of equipment that sells for $250 and has discount rates of 15% and 10%. Determine the single discount that could be used to replace both.

37. The list price of a motor is $325. What is its selling price if discounts of 10% and 2½% are used?

38. A casting weighed 32.5 lb before finishing and 30 lb 14 oz after finishing. What percent of the original weight remained?

39. The speed of a motor drops from a no load value of 1750 rpm to a full load value of 1675 rpm. What is its speed regulation?

40. The secondary voltage of a transformer drops from 240 to 236 as load is applied. What is its voltage regulation?

41. It takes 6.5 hr for a car to travel a certain distance at an average speed of 35 mph. What time would be needed at an average speed of 44 mph?

42. The passage of heat through a material generally varies directly with the thickness. 75 calories of heat pass per second through a plate 0.015 in. thick. What thickness would pass 96 calories per second?

43. The quantity (lumens) of light passing through an aperture varies with the area of the aperture. 10 lumens of light pass through an aperture of area 0.75 sq in. How much light would pass through an aperture of area 0.63 sq in.?

44. The number of seconds exposure time required in a camera is proportional to the square of the f number. If $\frac{1}{25}$ sec is needed at f:4.5, what is needed at f:35?

45. The illumination on a surface is directly proportional to the intensity of the source and inversely proportional to the square of the distance from the source to the surface.

An enlarger lamp having an intensity of 72 cp produces an illumination of 8 ft-candles on the easel 3 ft away. What would be the illumination if the lamp were (*a*) 25 cp; (b) 72 cp but 2 ft away?

46. For a thin lens there is a direct proportion between the lengths of the object and image and their respective distances from the lens. Find the items missing from the following table:

Object distance	*Object length*	*Image distance*	*Image length*
?	6 ft	6 in.	1 in.
15 in.	3 in.	6 in.	?
4.4 cm	2.68 cm	?	4.37 in.
$8\frac{1}{8}$ in.	?	$10\frac{1}{32}$ in.	$1\frac{1}{16}$ in.

47. Kodabromide 11×14 double weight is sold at the following prices:

Number of sheets	Net price	List price
10	$ 0.85	$ 1.30
25	3.77	5.80
250	15.96	24.55

(a) By what percent does each list price differ from each net price?

(b) What percentage saving is possible if paper is bought in 250-sheet packages as compared with the 10-sheet cost?

48. A photographic lamp sells to retailers at $3.39, less 20%. At what price must it be sold to gross 40%?

49. A lens costs the dealer $310. He wishes to gross 30%; find his selling price. If at a sale he reduces the marked retail price by 25%, what is his final dollar profit?

50. A bank offers small loans as follows: You borrow $100 and pay it back in 12 equal installments of $9.75. What is the percent interest?

Chapter 3

THE FUNDAMENTAL OPERATIONS IN ALGEBRA

1. Introduction. Arithmetic consists essentially of the operations of addition, subtraction, multiplication, and division, wherein numbers are used to express ideas and to save time and effort in the solutions of problems. By use of these operations or various combinations of them, many problems can be solved, but the solutions of a great many more problems require a knowledge of mathematics other than arithmetic, and therefore the technical student needs a thorough knowledge of algebra in order to cope with them.

Algebra is a continuation of arithmetic and makes use of the same signs and symbols, but also new signs, new symbols, and new methods are used to make for further simplification. General methods are developed that make it possible to solve whole classes of problems immediately, thereby reducing much of the work to routine.

Whereas arithmetic uses numbers to express ideas, algebra uses both numbers and letters to represent quantities whose values may or may not be known. For example, in electrical terminology, currents are represented by the letters I or i; resistances by R or r; and voltages by E, e, V, or v. The altitude or height of a triangle is usually represented by the letter h and the base of the triangle by the letter b. Letters or symbols of this type that are used to represent quantities in a general way are known as *general numbers* or *literal numbers.*

The literal-number idea is a very important one. By it we are enabled to express the various laws and facts relating to engineering in a simple concise mathematical form that is easily understood and interpreted, instead of writing them out in long wordy statements that are likely to be quite confusing. For example, Ohm's law for the direct-current electrical circuit states that the current in any part of a circuit is equal to the voltage across that part of the circuit divided by the resistance of that part. This makes a long confusing statement, whereas in mathematical form it is written $I = V/R$, where I represents the current, V the voltage, and R the resistance of the circuit. This simplified form of the long statement in words is known as a formula and is applicable to any direct-current circuit. When definite values are substituted for the letters, then the resulting expression applies only to a particular circuit. In the expression $C = \pi d$, the circumference of a circle is given as the product of the diameter D and a constant π. The expression with the letters applies to

any circle, but if figures are substituted for the letters, the resulting expression applies to one circle only.

In the above formulas, the first letter of each word was used to designate the quantities, and this practice is a common one in the engineering field. However, it may not always be possible or desirable to follow this practice and, in such cases, any letters may be used provided their meaning is defined properly. As a matter of fact, letters from the Greek alphabet are used quite extensively in mathematics as well as letters from the English alphabet.

It should be noted that there is a difference between the way in which a number in arithmetic is read and the way in which an algebraic expression containing letters is read. Thus, in arithmetic the number *36 means "3 times 10, plus 6,"* not "3 times 6," but the similar algebraic expressions *$3a$ and $4b^2$ mean "3 times a" and "4 times b^2,"* respectively. Example 3-1 illustrates the difference between the algebraic solution and the arithmetic solution of a problem.

EXAMPLE 3-1. In a classroom of 24 students there are two times as many men as women. How many of each are there?

Arithmetic solution:

	Some number	= number of women;
	2 times this number	= number of men.
Then	3 times this number	= total number of students,
or	3 times this number	= 24.
Therefore,	the number	$= \dfrac{24}{3} = 8$ women,
and	2 times the number	= 16 men.
To check:	8 + 16	= 24.

Algebraic solution:

Let	x	= number of women.
Then	$2x$	= number of men.
Therefore,	$2x + x$	= total number of students,
or	$3x$	= 24.
Then	x	= 8 women,
and	$2x$	= 16 men.
Check:	$8 + 16$	= 24.

It is obvious at once that the arithmetic solution is awkward and confusing whereas the algebra solution is simple and much easier to understand. The letter x has been used in the algebra solution to represent *some number* in the arithmetic solution, but any other letter of the alphabet could have been used just as well. In general, *the first letters of the alphabet, a, b, c, d, etc., are used to denote known values (usually called constants) and the last letters, u, v, w, x, y, and z are used to denote unknown values.*

2. Definitions. A number of definitions are important to the study of algebra and should be learned.

An *algebraic expression* is one that represents an idea by means of the signs and symbols of algebra. If it consists entirely of numerals and signs, such as $6 + 5 - 3$, it is a *numerical expression.* If it consists of letters, numerals, and signs, such as $3ab + 2cd$, it is a *literal expression.*

A *term* is a part of an algebraic expression that is not separated by a plus or a minus sign. Thus, in the expression $7ab + 2bc$, there are two terms, $7ab$ and $2bc$.

An algebraic expression that contains only one term is a *monomial;* one that contains more than one term is a *polynomial.* A *binomial* is a polynomial of two terms; a *trinomial* is a polynomial of three terms.

$5xy$ and $-4ac$ are monomials.
$3xy - 7ab$ is a binomial and a polynomial.
$2x + 3y - 4z$ is a trinomial and a polynomial.

If two or more numbers are multiplied together to form a term, any one of them or combination of them is a *factor* of the term. In the term $3xyz$; 3, x, y, z, $3x$, $3y$, $3z$, xy, and yz are all factors of the term $3xyz$. A *prime factor* is a factor that can be divided only by itself and 1.

A *coefficient* is a multiplier of a term. It is generally taken as the numerical part of the term, although it may include letters as well. In the algebraic term $4ay$, 4 is the coefficient of ay, $4a$ is the coefficient of y, and $4y$ is the coefficient of a. If the expression does not contain a numerical coefficient, then the numerical coefficient is understood to be 1. Thus, xyz means the same as $1xyz$.

The *lowest common multiple* of two or more expressions is the product of all their prime factors, with each factor being taken the largest number of times that it occurs in any one expression. The lowest common multiple of $3ax$, $6b$, and $12bxy$ is $12abxy$.

An *exponent* is a figure or letter placed a little above and to the right of a quantity, and it indicates the number of times that the quantity is taken as a factor. The quantity is called the *base* and the exponent is called the *power.* For example, x^2 indicates that x is to be taken two times as a factor; x^3 indicates that x is to be taken three times as a factor. x^2 is read "x squared," or "x to the second power"; x^3 is read "x cubed" or "x to the third power." The product of $a \times a \times a \times a \times a$ is written a^5 and is read "a to the fifth power." It is important that an exponent be distinguished from a coefficient. $3x$ means $x + x + x$, whereas x^3 means $(x)(x)(x)$.

If all the factors of a quantity are equal, one of the factors is called a *root* of the quantity. One of the two equal factors is a *square root;* one of the three equal factors is a *cube root;* one of the four equal factors is a *fourth root,* and so on. The symbol used to indicate that a root is to be taken is $\sqrt{\ }$, and is called a *radical sign.* If no index figure is written in the

radical sign, the second or square root is understood. Otherwise, an index figure is written in the radical to indicate the root. Thus,

$\sqrt{a}$ means the square root of a and $\sqrt[3]{a}$ means the cube root of a.

$\sqrt{a}$ is also written $a^{1/2}$ and is read "a to the one-half power." Likewise $\sqrt[3]{a}$ could be written $a^{1/3}$ and read "a to the one-third power." Similarly,

$$\sqrt[3]{a^2} = a^{2/3}; \quad \sqrt[4]{a} = a^{1/4}; \quad \sqrt[4]{a^2} = a^{2/4} = a^{1/2}.$$

Similar terms are terms containing the same letters with the same exponents but not necessarily with the same coefficients. *Dissimilar terms* are terms containing different letters or the same letters with different exponents.

$4x$ and $6x$ are similar terms.
$4x^2$ and $7x^2$ are similar terms.
$4x$ and $4x^2$ are dissimilar terms.
$4a$ and $4b$ are dissimilar terms.

The *degree* of a term is determined by adding the exponents of its literal factors together. Thus, x^2 is a second-degree term and ax^2y is a fourth-degree term. The degree of a polynomial is the highest degree of any of its terms. The *reciprocal* of a term is 1 divided by that term.

3. Positive and negative numbers. In arithmetic we dealt with only positive numbers, but in algebra the numbering system is extended to include negative numbers. We used the signs of addition and subtraction in arithmetic (+ and −) to indicate an operation to be performed between numbers, but the numbers themselves were all considered as positive. We did not deal with any numbers considered the opposite of positive. The idea of a negative number is necessary in algebra because it is impossible to handle some algebraic expressions without this idea.

We may represent arithmetical numbers in an ascending scale as shown in Fig. 3–1, and reduce the operations of addition and subtraction to

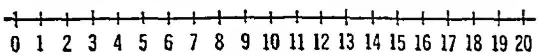


Fig. 3–1.

counting along this scale. 2 is added to 1 by starting at 1 and counting 2 units to the right, thus giving 3. Now to subtract 4 from 5, start at 5 and count to the left 4 units, thus reaching 1. This direction is considered as negative. If we want to subtract 5 from 4, we find we cannot do it on this scale, or, in other words, it cannot be done in arithmetic. But if the scale is constructed so that there are numbers on each side of the zero point, then the subtraction of 5 from 4 can be completed. The scale will appear as shown in Fig. 3–2, and subtracting 5 from 4 will give −1, when

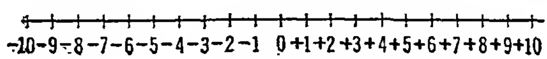


Fig. 3–2.

counting 5 units to the left from 4. As will be seen, numbers to the left of the zero have a minus sign in front of them and numbers to the right of the zero have a plus sign in front of them. The value of any number, without regard to the sign before it, is called its *absolute value.* Thus, both $+2$ and -2 have the same absolute value. If two numbers of the same absolute value but of different signs are taken together, they cancel each other and leave zero. Thus, $+2 -2 = 0$. The zero is neither positive nor negative but may be defined as the result of subtracting one number from itself.

It is evident therefore that we now have two kinds of quantities to deal with, positive and negative. A positive quantity is designated by the plus (+) sign, but this sign is generally omitted provided the omission causes no confusion to arise. Therefore, if no sign precedes a quantity, it is understood to be positive. A negative quantity is always preceded by a minus (−) sign. *This minus sign therefore has a double function: In arithmetic it is used to indicate the process of subtraction; in algebra it is used to indicate a negative number.*

There are many examples of the uses of negative numbers in the engineering field, and probably one of the most common uses is in the reading of temperatures. The temperature of melting ice, which is commonly known as freezing temperature, has been chosen as the zero point on the centigrade thermometer. But temperatures fall below this freezing point in a good many instances and these would be below the zero point. Therefore, they are indicated by a negative number, since all numbers above the zero point are positive. The total change in temperature would be the number of unit degrees between any two points on the scale. Thus, a change in temperature from $+2$ degrees to -2 degrees would actually mean a change of 4 degrees. This can be seen by referring to Fig. 3–2, in which 4 units will be found between $+2$ and -2.

4. Addition. The addition and subtraction of quantities are given a wider range by the use of the negative numbers just discussed. Addition does not always mean an increase nor does subtraction always mean a decrease. The following rules for addition result from the inclusion of negative numbers.

1. In adding two algebraic numbers with like signs, add their absolute values and give the result the common sign.

2. In adding two algebraic numbers with opposite signs, find the difference of their absolute values and give the result the sign of the larger number.

3. In adding more than two algebraic numbers with both positive and negative signs, find the sum of the positive numbers, then the sum of the negative numbers, and apply Rule 2 to this result.

The additions in Example 3–2 illustrate the foregoing rules.

Example 3-2:

$$\begin{array}{rrrrrrr}
+4 & +4 & +3 & +\ 5a & +6xy & +8xy & -\ 7ab \\
+2 & -3 & -5 & +\ 6a & -7xy & -6xy & +\ 2ab \\
 & & & & & +5xy & -\ 8ab \\
 & & & & & -3xy & +\ 3ab \\
\hline
+6 & +1 & -2 & +11a & -\ xy & +4xy & -10ab
\end{array}$$

Similar terms in algebra are added directly and the sum of any number of similar terms is the same regardless of the manner in which the terms are grouped.

Dissimilar terms cannot be added directly; the addition is simply indicated by the operational sign between them.

Example 3-3. Perform the following additions:

$$6x + 5x + \frac{2}{3}x - \frac{1}{3}x; \quad 14a + 15b; \quad 3a - 4c + 2a.$$

Solution:

$$6x + 5x + \frac{2}{3}x - \frac{1}{3}x = 11x + \frac{1}{3}x = 11\frac{1}{3}x = \frac{34}{3}x.$$

$$14a + 15b = 14a + 15b.$$

$$3a - 4c + 2a = 5a - 4c.$$

Example 3-4. Find the sum of the following expressions:

$$2a - 6b + 4c; \quad 5a - 3c; \quad b - a; \quad \text{and} \quad -a - 2b - 3c + d.$$

Solution: Similar terms are first written in columns and the algebraic sum of each column is then found.

$$\begin{array}{rrrrr}
 & 2a & -\ 6b & +\ 4c & \\
 & 5a & & -\ 3c & \\
 & -\ a & +\ \ b & & \\
 & -\ a & -\ 2b & -\ 3c & +\ d \\
\hline
\text{Sum} = & 5a & -\ 7b & -\ 2c & +\ d
\end{array}$$

5. Subtraction. Subtraction in algebra has a wider meaning than in arithmetic because of the inclusion of negative numbers. Now, it is not necessary that the minuend be larger than the subtrahend nor is it necessary that both be positive. Either or both may be negative. *Subtracting a term in algebra is the same as adding it with its sign changed.* Therefore, *to subtract one term from another, change the sign of the term to be subtracted, and then add.* The change in sign is indicated by writing the new sign under the old sign. To subtract polynomials, place like terms in columns and then subtract the terms in the columns.

Example 3-5. Perform the following subtractions: $+6x$ from $+15x$; $-6x$ from $+15x$; $-4E$ from $-20E$; $+4R$ from $-20R$; the polynomial $+3ax^2 - 4IR + 6cz$ from the polynomial $+8ax^2 + 12IR - 2cz$.

Solution:

$$
\begin{array}{r}
+15x \\
+\ 6x \\
- \\
\hline
+\ 9x
\end{array}
\qquad
\begin{array}{r}
+15x \\
-\ 6x \\
+ \\
\hline
+21x
\end{array}
\qquad
\begin{array}{r}
-20E \\
-\ 4E \\
+ \\
\hline
-16E
\end{array}
\qquad
\begin{array}{r}
-20R \\
+\ 4R \\
- \\
\hline
-24R
\end{array}
\qquad
\begin{array}{rrr}
+8ax^2 & +12IR & -2cz \\
+3ax^2 & -\ 4IR & +6cz \\
- & + & - \\
\hline
+5ax^2 & +16IR & -8cz
\end{array}
$$

6. Signs of grouping. Terms that are to be affected by the same operation and thus to be considered as a single quantity are generally grouped together by means of the symbols for parentheses, ( ), brackets, [ ], or braces, { }, and, since all these symbols have the same effect, we shall use the word *parentheses* to mean any one of them. Parentheses are used to indicate that the operations enclosed by them should be performed first; and the positive or negative sign in front of the parentheses affects all terms within the parentheses in accordance with the following rules:

1. When parentheses are preceded by a positive sign, they can be removed and the sign of each term enclosed by the parentheses will remain unchanged.

2. When parentheses are preceded by a negative sign, they can be removed if the sign of each term enclosed by the parentheses is changed.

Example 3-6. Remove the parentheses and simplify each of the following:

(a) $3x + (5y - 2z)$.
(b) $3x - (5y - 2z)$.

Attention is called to the fact that the sign of $5y$ in each case is $+$.

Solution:

(a) $3x + (5y - 2z) = 3x + 5y - 2z$.
(b) $3x - (5y - 2z) = 3x - 5y + 2z$.

Oftentimes, several pairs of parentheses are found in one problem. In this case, the inner pair is removed first, then the next outer pair, and so on until all have been removed. The above rules must be applied for each removal.

Example 3-7. Simplify the following expression:

$$8R - [x - \{5a + (3b - 2a) - 4b + 3x\} + 6R].$$

Solution:

$$
\begin{aligned}
&8R - [x - \{5a + (3b - 2a) - 4b + 3x\} + 6R] = \\
&8R - [x - \{5a + 3b - 2a - 4b + 3x\} + 6R] = \\
&8R - [x - 5a - 3b + 2a + 4b - 3x + 6R] = \\
&8R - x + 5a + 3b - 2a - 4b + 3x - 6R = \\
&\qquad\qquad 2R + 2x + 3a - b.
\end{aligned}
$$

7. Insertion of grouping signs. There are occasions when it becomes necessary to insert parentheses instead of removing them. If it is required to enclose terms in parentheses preceded by a plus sign, the sign of each

term remains the same as before the enclosure. If it is required to enclose terms in parentheses preceded by a minus sign, the sign of each term so enclosed must be changed.

EXAMPLE 3–8. Enclose the last two terms of (a) in parentheses preceded by a plus sign and of (b) in parentheses preceded by a minus sign.

(a) $a + b + c - d$.
(b) $m - n + r - s$.

Solution:

(a) $a + b + c - d = a + b + (c - d)$.
(b) $m - n + r - s = m - n - (-r + s)$.

EXERCISE 3–1

Find the sum in each of the following problems:

1. $4R$; $7R$; $-15R$; $20R$.
2. $14a-6b+3c$; $5d+2b-3a$; $-6a+7b-3d$; $8c+5a-4b-13c+2d$.
3. $25EI$; $-12EI$; EI; $-6EI$.
4. $4x-7y+2z$; $-5x-6z+3y$; $12z+4x-2y$; $8y+15x+14z-10y$.
5. $-12a + 15b + 2d$; $5a + 8b - 10c$; $a + 14b - 5d + 2c$; $6d + 4a - 3d - 5b$; $13c - 14b + 16d - 12c$.
6. $4 - 2\sqrt{R^2 + x^2} - z$; $5\sqrt{R^2 + x^2} - 2r + 2z$; $5 - \sqrt{R^2 + x^2} + 5r - 4z$.
7. $12x^{3/4} + 15y^{2/3} - 15z^{3/5}$; $6z^{3/5} - 9y^{2/3} - 3x^{3/4}$; $21y^{2/3} + 25x^{3/4} + 16z^{3/5}$; $14x^{3/4} + 12z^{3/5} - 10y^{2/3}$.
8. $4x^{1/2}-6y^{1/2}-z$; $6z-5y^{1/2}-6x^{1/2}$; $12y^{1/2}-4z-2x^{1/2}$; $8y^{1/2}-7x^{1/2}-10z$.
9. $17y^2 - 12x^3 + 16z^4$; $-8z^4 + 4x^3 - 10y^2$; $15x^3 + 14z^4 - 4y^2$; $-5y^2 - 20z^4 - 3x^3$.
10. $6a^3b - 9a^2b^2 - 4ab^3$; $-12a^3b + 8ab^3 - 6a^2b^2$; $12a^2b^2 - 2a^3b + 7ab^3$; $9a^3b + 6ab^3 - 8a^2b^2$.
11. $-7ax - 5a\sqrt{x} + 12y$; $5ax + a\sqrt{x} - 8y$; $-ax + 9a\sqrt{x} - y$; $4ax - 3a\sqrt{x} + 5y$; $-ax + a\sqrt{x} + y$.
12. $3x^6 + 5x^5y\sqrt{2} + 2x^4y^2$; $4y^6 - 7xy^5 + 3x^2y^4 - 9x^3y^3$; $7x^3y^3\sqrt{2} - 3x^2y^4 - 3y^6$; $6x^5y\sqrt{2} - 10x^4y^2 + 5x^3y^3 - 12x^2y^4$.
13. $a^2 + \sqrt{x} - 10 + 2\sqrt{ax}$; $2a^2 - 3\sqrt{x} + 20 + \sqrt{ax}$; $-5a^2 - 3\sqrt{x} + 30 - 5\sqrt{ax}$; $-4a^2 + 2\sqrt{x} + 1 + 12\sqrt{ax}$; $-25a^2 - 25\sqrt{x} - 25 - 25\sqrt{ax}$.
14. $3x^4\sqrt{a} - 2x^3y\sqrt{b} + 7x^2y^2 + 6y^4$; $8y^4 - 2xy^3 - 9x^2y^2$; $10xy^3 + 9x^2y^2 + 3x^3y\sqrt{b}$; $4x^4\sqrt{a} + 3x^3y + 12xy^3 - 9y^4$.

Perform the following subtractions:

15. $2\frac{1}{2}b^2$ from $\frac{11a^2}{2} - \frac{15b^2}{4}$.
16. $(0.3x^3 - 5 + 6x^2 + 6x)$ from $6x^3 + 0.4x^2 - 3x + 2$.
17. $\left(\frac{5a}{6} - \frac{7b}{8}\right)$ from $\frac{2a}{3} - \frac{4b}{5}$.
18. $(4x - 2y + 8c - 3)$ from $8x + y - 7c$.

Simplify the following:

19. $a + b - (c - d)$.
20. $n + (2n - m) - (3m - n) + m$.
21. $p - (q - r - s) + (s + q) - (q + r)$.
22. $5x - [2b - \{3a + (2a - b) - a\} + 2x]$.
23. $4IR - [5IX + 2E - (2IR - 4IX) + (-9IR + 6 - 8E)]$.
24. From $8IR + 3E - 6IX$ subtract $2E - 12IR - 16IX$.
25. By how much does 0 exceed $a^2 - b^2$? By how much does 1 exceed $x^2 - 2xy + y^2$?
26. $1 - x - \{1 - [x - (x - 1) - (2 - x)] - [1 - x - (x + 1)]\}$
27. $-[-5ax - (6by - 5cz) + 2cz - (7ax - \{ax - 3cz\} + 2by)]$.
28. $ab - [5 - x - (c - b - 2ab - x)] + [b - (x + 4 - c)]$.
29. $7a + 2b - [3a - c - (2b - 3c)]$.
30. $9a - 4c - \{[3b - 4c + 5a] - 3b\}$.
31. $x - 5[2x - 3(y - 2z) - (4x + y - 2z)]$.
32. $2a - 3(a + b) - 0.5(2a + 0.2b)$.
33. Enclose the last three terms of each of the following in positive parentheses:

    (a) $E + e - ir + ix$.

    (b) $x^3 - 3x^2y + 3xy^2 - y^3$.
34. Enclose the last three terms of each of the parts of Problem 33 in negative parentheses.

8. Multiplication. The very same principles of multiplication that are used in arithmetic are also used in algebra and the same notation is employed. The terms *multiplicand, multiplier,* and *product* still mean exactly the same. Multiplication is indicated in arithmetic by the symbol (×) or by the dot (·) but these symbols are not used to any extent in algebra because they are likely to cause confusion. The symbol (×) may be confused with the letter x and the dot may be confused with a decimal point. Therefore, it is best always to use the parenthesis to indicate a multiplication in algebra. However, the product or multiplication of two different letters, such as a and b, is generally indicated by writing the letters in juxtaposition with no sign between them; thus, ab. The actual multiplication cannot be performed until numerical values are substituted for the letters.

Since we are dealing with negative as well as positive quantities, it becomes necessary to extend the principles of arithmetic in order to determine the sign of a product when combinations of these quantities are multiplied. Following are the rules for algebraic multiplication:

1. The product is positive when two terms having like signs, either positive or negative, are multiplied.

2. The product is negative when two terms having unlike signs are multiplied.

3. The product is positive when any number of positive terms are multiplied.

4. The product is positive when an even number of negative terms are multiplied and negative when an odd number of negative terms are multiplied.

5. The coefficient of the product is equal to the product of the coefficients of the several factors multiplied.

EXAMPLE 3–9. Multiply the following:

(a) $(4a)(3b)(2c)$.
(b) $(-3a)(-2b)(-d)(-3c)$.
(c) $(-2a)(-4b)(-6c)$.
(d) $(6x)(7y)$.
(e) $(-4y)(3z)$.

Solution:

(a) $(4a)(3b)(2c) = +24abc$.
(b) $(-3a)(-2b)(-d)(-3c) = +18abcd$.
(c) $(-2a)(-4b)(-6c) = -48abc$.
(d) $(6x)(7y) = 42xy$.
(e) $(-4y)(3z) = -12yz$.

9. Exponents in multiplication. In Article 2, an exponent was defined as indicating the number of times a quantity (called the base) is taken as a factor. Thus, $x^5 = (x)(x)(x)(x)(x)$.

Now if $$a^3 = (a)(a)(a)$$
and $$a^2 = (a)(a),$$
then $$(a^3)(a^2) = (a)(a)(a)(a)(a) = a^5,$$
or $$(a^3)(a^2) = a^{3+2} = a^5.$$

Therefore, we have the rule: *When two or more terms having the same base are multiplied together, the product is found by raising the base to a power equal to the sum of the exponents of the several terms.*

EXAMPLE 3–10.

(a) $(a^3)(a^4) = a^{3+4} = a^7$.
(b) $(x^2)(x^3) = x^{2+3} = x^5$.
(c) $(5^2)(5^3) = 5^{2+3} = 5^5$.
(d) $(e^x)(e^y) = e^{x+y}$.
(e) $(a)(a) = a^{1+1} = a^2$.

Solution: When an exponent is not indicated, it is understood to be 1. Thus in the solution of Problem (e), Example 3–10, the two factors, $(a)(a)$, each have an exponent of 1, giving a^{1+1} or a^2 as the product.

It should be noted that exponents can be added only when the terms have the same base. If the bases are not the same, then the multiplication can be indicated only.

Thus, $(a^2)(a^3) = a^{2+3} = a^5.$

But $(a^2)(b^3) = a^2b^3.$

Therefore, it is evident that the law for exponents can be stated in the general form

$$(x^m)(x^n) = x^{m+n}$$

where m and n are general numbers and x is not equal to zero.

10. Multiplication of monomials. To multiply monomials, the product of the numerical coefficients is taken, and this product in turn is multiplied by the product of the literal factors, the law of exponents being used where it applies. The proper sign must be attached to the result in accordance with the rules of Article 8.

EXAMPLE 3–11. Find the products in each of the following:

(a) $(3ab^2)(2a^2b)$.
(b) $(-3x^2y^3)(4xy^2)$.
(c) $(-5x^3y^2)(-3x^2y^4)$.

Solution:

(a) $(3ab^2)(2a^2b) = 6a^3b^3$
(b) $(-3x^2y^3)(4xy^2) = -12x^3y^5.$
(c) $(-5x^3y^2)(-3x^2y^4) = +15x^5y^6.$

11. Multiplication of polynomials by monomials and polynomials. To multiply a polynomial by a monomial, each term of the polynomial is multiplied by the monomial and the resulting terms are written in succession with their proper signs. Thus the product of the monomial a and the polynomial $(b + c + d)$ is $a(b + c + d) = ab + ac + ad$.

To multiply a polynomial by a polynomial, each term of the multiplicand is multiplied by each term of the multiplier and the algebraic sum of these separate products is found.

EXAMPLE 3–12. Multiply $2x + y$ by $x - 2y$.

Solution:

$2x + y$	multiplicand
$x - 2y$	multiplier
$2x^2 + xy$	multiplying $2x + y$ by x
$\quad - 4xy - 2y^2$	multiplying $2x + y$ by $-2y$
$2x^2 - 3xy - 2y^2$	algebraic sum of separate products

It is evident from Example 3–12 that multiplying one polynomial by a second polynomial is equivalent to multiplying the first polynoimal by each of the monomials that make up the second polynomial and finding the sum of the resulting terms. There are occasions when the addition of

some of the terms will give zero, and this addition then does not appear in the product.

EXAMPLE 3–13. Multiply $a^2 - ab + b^2$ by $a + b$.

Solution:

$$\begin{array}{r}
a^2 - ab + b^2 \\
a + b \\
\hline
a^3 - \cancel{a^2b} + \cancel{ab^2} \qquad \\
+ \cancel{a^2b} - \cancel{ab^2} + b^3 \\
\hline
a^3 \qquad\qquad + b^3
\end{array}$$

There are certain special cases in multiplication in which the product may be written at sight.

1. The square of the sum of two quantities is equal to the square of the first plus two times the product of the first and second plus the square of the second.

$$(a + b)^2 = a^2 + 2ab + b^2.$$
$$(x + y)^2 = x^2 + 2xy + y^2.$$
$$(3 + 2)^2 = 3^2 + (2)(3)(2) + (2)^2 = 9 + 12 + 4 = 25.$$

2. The square of the difference of two quantities is equal to the square of the first minus two times the product of the first and second plus the square of the second.

$$(a - b)^2 = a^2 - 2ab + b^2.$$
$$(x - y)^2 = x^2 - 2\,xy + y^2.$$
$$(5 - 3)^2 = 5^2 - (2)(5)(3) + 3^2 = 25 - 30 + 9 = 4.$$

3. The square of any trinomial is equal to the algebraic sum of the squares of the terms and two times the product of each term by each term that follows it.

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

4. The product of the sum and difference of two quantities is equal to the difference of their squares.

$$(x + y)(x - y) = x^2 - y^2.$$
$$(5 + 3)(5 - 3) = 5^2 - 3^2 = 25 - 9 = 16.$$

Attention is called particularly to the expansions shown in 1 and 2. A common error made by students is to expand $(x + y)^2$ into $x^2 + y^2$, and $(x - y)^2$ into $x^2 - y^2$. It can be easily shown that these are incorrect by reference to the expansions of $(3 + 2)^2$ and $(5 - 3)^2$. $(3 + 2)^2$ is obviously 25, but $3^2 + 2^2$ is 13. Similarly, $(5 - 3)^2$ is 4, but $5^2 - 3^2$ is 16.

12. The binomial theorem. An expression, called the binomial theorem, has been developed that is very useful in raising a binomial to any power. By actual multiplication we find that

$$(x + y)^1 = x + y,$$
$$(x + y)^2 = x^2 + 2xy + y^2,$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Evidently, the expansion of $(x + y)^n$, where n* may take any positive value, is done as follows:

1. The first term is x^n.

2. As the expansion progresses from any term to the next following term the exponent of x decreases by 1 and the exponent of y increases by 1. The expansion is complete when the first term reaches a zero exponent and the second term reaches the desired exponent. (Since $y^0 = 1$, it is omitted from the first term of the expansion, and since $x^0 = 1$, it is omitted from the last term of the expansion.)

3. If the numerical coefficient of any term is multiplied by the exponent of x in that term and the product divided by one more than the exponent of y in that term, the result will be the numerical coefficient of the next succeeding term.

4. The exponents of x and y in any term add up to n.

From these observations, the following formula has been deduced:

$$
\begin{aligned}
(x+y)^n = x^n + nx^{n-1}y &+ \frac{(n)(n-1)}{2!}x^{n-2}y^2 \\
&+ \frac{(n)(n-1)(n-2)}{3!}x^{n-3}y^3 + \frac{(n)(n-1)(n-2)(n-3)}{4!}x^{n-4}y^4 \\
&+ \frac{(n)(n-1)(n-2)(n-3)(n-4)}{5!}x^{n-5}y^5 \ldots + y^n
\end{aligned}
$$

where 2! is called factorial 2 and equals (1)(2), 3! is called factorial 3 and equals (1)(2)(3), 4! is called factorial 4 and equals (1)(2)(3)(4), and 5! is called factorial 5 and equals (1)(2)(3)(4)(5).

If the second term of the binomial to be expanded is negative, then the signs of the terms in the expansion alternate positive and negative, beginning with positive for the first term.

The binomial theorem can be applied to any complicated form that can be put in the binomial form by the insertion of parentheses. Thus the expansion of $[a + b - 3]^3$ can be performed by first putting it in the form $[(a + b) - 3]^3$. Now $(a + b)$ is treated as a single term and the expansion can be performed by the binomial theorem.

EXAMPLE 3–14. Expand $[(a + b) - 3]^3$.

Solution:

$$
\begin{aligned}
[(a + b) - 3]^3 &= (a + b)^3 - (3)(a + b)^2(3) + 3(a + b)(3)^2 - (3)^3, \\
&= a^3 + 3a^2b + 3ab^2 + b^3 - 9(a^2 + 2ab + b^2) + 27(a + b) - 27, \\
&= a^3 + 3a^2b + 3ab^2 + b^3 - 9a^2 - 18ab - 9b^2 + 27a + 27b - 27.
\end{aligned}
$$

EXERCISE 3–2

Perform the following multiplications:

1. $(3x)(2y)(5z)$.
2. $(-4a)(-3b)(-2c)(-d)$.
3. $(6x)(-7y)(2z)$.
4. $(c - 6)(c + 6)$.
5. $(c + d)(c - d)$.
6. $(ab + 7)(ab - 7)$.

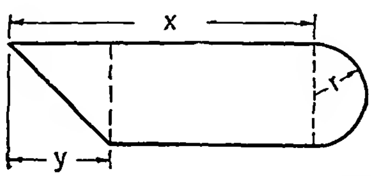
* A negative exponent is treated by writing the expression as a reciprocal.

7. $(2a^2 + 3b^2)(2a^2 - 3b^2)$.
8. $(x^n - y^n)(x^n + y^n)$.
9. $(-4I^2R)(8IZ)(-6ZR^2)$.
10. $(0.5i^2e)(3ie^2)(-0.05e^2rx)(0.1rx^2)$.
11. $(a + jb)(a + jb)$.
12. $(x - 4y + 6z)(2x + y + 2z)$.
13. $(3x + 5)(4x - 3)$.
14. $(5x + 10)(4x + 6)$.
15. $(4a^2 - 1)(a^2 + 6)$.
16. $(a^2c^2 + 4b)(2a^2c^2 - 3b)$.
17. $(x^{n+1} + y^{m-1})(x^{n+1} - y^{m-1})$.
18. $(9x^m + 6y^n)(9x^m - 6y^n)$.
19. $[(x + y) + 5][(x + y) - 5]$.
20. $(x^2 + \frac{1}{2})^3$.
21. $(x^4 - m^2 + 1)(x^4 + m^2 + 1)$.
22. $(a + b + c)(a - b - c)$.
23. $(0.5m^4 - 0.4m^3n + 1.2m^2n^2 + 0.8mn^3 - 1.4n^4)(0.4m^2 - 0.6mn - 0.8n^2)$.
24. $(14c^4 - 18c^3d + 15c^2d^2 - 8cd^3 + \frac{1}{3}d^4)(12c^5 + \frac{1}{3}c^4d - 10c^3d^2 - \frac{1}{2}c^2d^3)$.
25. $(0.2x^2 - 3xy - y^2)(-0.02x^2 + xy + 0.1y^2)$.
26. $(7y - 0.5y^2 + 4)(0.2 - 4y + 0.7y^2)$.
27. $(3a + 2b - 4c)(3a + 2b + 4c)$.
28. $(x^2 + 2x - 1)(x^2 - 2x + 1)$.
29. $[(x - y) + a]^4$.
30. $[(x + y) + (a + b)]^3$.
31. $(x^2 + xy + y^2)(x^2 - xy + y^2)$.
32. $(2a - 9b)^3$.

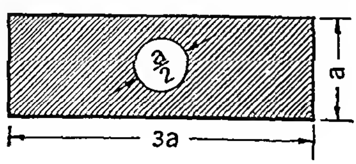
33. The area (A) of a trapezoid is equal to the product of the altitude (h) and one half the sum of the parallel sides, a and b. Thus, $A = \frac{1}{2}(a + b)h$. Find the area of the trapezoid whose altitude is y inches and whose parallel sides are $(y + 2)$ inches and $(y + 6)$ inches.

34. The area (A) of a triangle is equal to one half the product of its base (b) and altitude (h). Thus, $A = \frac{1}{2}bh$. Find in square feet the area of the triangle whose base is x in. and whose altitude is $(x - 4)$ in.

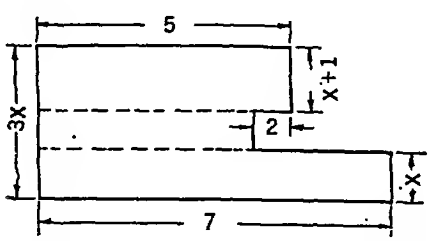
35. Find the total area of the following figure by adding the areas of the triangle, the rectangle, and the semicircle. The area (A) of a rectangle is the base (b) multiplied by the altitude (h), or $A = bh$ and the area (A) of a semicircle is $\frac{\pi r^2}{2}$.



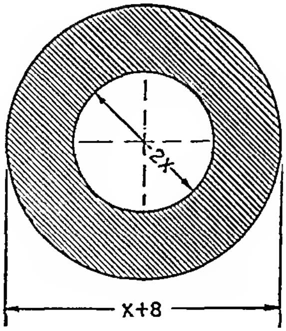
36. Find the area of the shaded portion in the figure below.



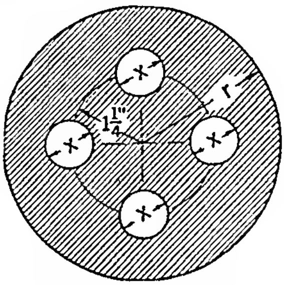
37. Find the total area.



38. Find the shaded area.



39. Find the shaded area.



13. Division. In multiplication two quantities are given and their product is required. The inverse process of finding one quantity when the other quantity and their product is given is called division. The terms dividend, divisor, and quotient are used and have the same meaning as in arithmetic: the *dividend* is the quantity to be divided; the *divisor* is the quantity by which the division is performed; and the *quotient* is the result. The dividend corresponds to the product in multiplication, the divisor to the multiplier, and the quotient to the multiplicand. Therefore, the quotient is defined as that quantity which when multiplied by the divisor produces the dividend.

14. Division of monomials. The following quotients are obtained by the principles of multiplication and the definition of a quotient:

$$(+2x)(+3y) = +6xy; \text{ therefore, } (+6xy) \div (+3y) = +2x.$$
$$(+2x)(-3y) = -6xy; \text{ therefore, } (-6xy) \div (-3y) = +2x.$$
$$(-2x)(+3y) = -6xy; \text{ therefore, } (-6xy) \div (+3y) = -2x.$$
$$(-2x)(-3y) = +6xy; \text{ therefore, } (+6xy) \div (-3y) = -2x.$$

From these quotients, we obtain the following rules for division in algebra:

1. The sign of the quotient will be plus when the dividend and divisor have like signs and minus when they have unlike signs.

2. The numerical coefficient of the quotient will be equal to the numerical coefficient of the dividend divided by the numerical coefficient of the divisor.

15. Exponents in division. Since in multiplication the exponent of the product is the sum of the exponents of the terms, in division the exponent of the quotient will be the exponent of the dividend minus the exponent of the divisor. Therefore, we have the rule: *When two terms of the same base form a division, the quotient is found by raising the base to a power equal to the difference between the exponents of the dividend and divisor respectively.*

EXAMPLE 3–15. Perform the following divisions:

(a) $\dfrac{16x^5}{4x^2}$. (b) $\dfrac{25x^2y^3z}{5xy^3z^2}$.

Solution:

(a) $\dfrac{16x^5}{4x^2} = 4x^{5-2} = 4x^3.$

(b) $\dfrac{25x^2y^3z}{5xy^3z^2} = 5x^{2-1}y^{3-3}z^{1-2} = 5xy^0z^{-1} = \dfrac{5x}{z}.$

Note that the quantity $y^{3-3} = y^0$, and is dropped in the final answer since any quantity raised to the zero power is equal to 1. This is proved easily.

Since $\dfrac{y^3}{y^3} = y^{3-3} = y^0$, and $\dfrac{y^3}{y^3} = 1$,

then $y^0 = 1.$

Also it should be noted that the exponent of z is $1 - 2$ or -1, and since $z^{-1} = \dfrac{1}{z}$, the final answer is written $\dfrac{5x}{z}$.

If the letters in the dividend are not the same as the letters in the divisor, the division cannot actually be performed but is merely indicated. If some of the letters are the same but others are different, then the division can be performed only for those letters that are alike.

Thus $\dfrac{4xy}{3ab}$ is left in this form; but $\dfrac{4xy}{3ax} = \dfrac{4y}{3a}$.

From the foregoing it is evident that the law of exponents for division can be put in the general form

$$\frac{x^m}{x^n} = x^{m-n}$$

where m and n are general numbers and x is not equal to zero.

16. Division of a polynomial by a monomial. To divide a polynomial by a monomial, each term of the polynomial is divided by the monomial and the resulting terms are written in succession with their proper signs.

EXAMPLE 3–16. Divide

(a) $ay + by$ by y,
(b) $6a^2b^3 - 15abc + 7ab^2c^2$ by $3ab$.

Solution:

(a) $\dfrac{ay + by}{y} = \dfrac{ay}{y} + \dfrac{by}{y} = a + b.$

(b) $\dfrac{6a^2b^3 - 15abc + 7ab^2c^2}{3ab} = \dfrac{6a^2b^3}{3ab} - \dfrac{15abc}{3ab} + \dfrac{7ab^2c^2}{3ab} = 2ab^2 - 5c + \dfrac{7}{3}bc^2.$

17. Division of a polynomial by a polynomial. The division of one polynomial by another polynomial is performed in exactly the same manner as is long division in arithmetic.

First arrange the dividend and divisor in either descending or ascending powers of some common literal factor. Then divide the first term of the dividend by the first term of the divisor and write this result as the first term of the quotient. Multiply the complete divisor by the first term of the quotient, place this product under the dividend, and subtract. The remainder gives a new dividend by which to divide the first term of the divisor as before. Repeat this process until the first term of the divisor will not be contained in the first term of the dividend. Any remainder is placed over the divisor as a fraction to be added to the quotient already found.

EXAMPLE 3–17. Divide $2x^2 + 13xy + 21y^2$ by $x + 3y$.
Solution:

$$
\begin{array}{llrrrll}
 & & & \text{dividend} & & & \\
\text{divisor} & \underline{x + 3y\,|} & 2x^2 & +\ 13xy & +\ 21y^2 & \underline{|\,2x + 7y} & \text{quotient} \\
 & & -\ 2x^2 & +\ \ 6xy & & & \\
 & & & - & & & \\
\hline
 & & & +\ \cancel{7xy} & +\ \cancel{21y^2} & & \\
 & & & +\ \cancel{7xy} & +\ \cancel{21y^2} & & \\
 & & & - & - & & \\
\hline
\end{array}
$$

x, the first term of the divisor, divides into $2x^2$, the first term of the dividend, $2x$ times. Thus, $2x$ is the first term of the quotient. Multiplying $2x$ by the complete divisor, $x + 3y$, gives $2x^2 + 6xy$, which is written under the proper terms of the dividend and subtracted. The remainder $7xy$ is made complete by bringing down the next term ($21y^2$) of the dividend, giving $7xy + 21y^2$ as a new dividend. Dividing $7xy$, the first term of the new dividend by x, the first term of the divisor, gives $7y$ as the next term in the quotient. Multiplying $7y$ by the divisor, $x + 3y$, gives $7xy + 21y^2$, which is subtracted from the dividend. Since the subtraction results in zero and there are no more terms in the dividend to bring down, the division is complete. The division can be checked by multiplying the divisor by the quotient.

EXAMPLE 3–18. Divide $x^4 + 4x^2 + 1$ by $x + 1$.

Solution: When the dividend is arranged in descending powers of x, it is found that there is no term in x^3 and no term in x. Therefore, allowance is made for them by inserting zero in place of each.

$$
\begin{array}{l|lllll|l}
\text{divisor} & \text{dividend} & & & & & \text{quotient} \\
x+1 & x^4 & +\,0 & +\,4x^2 & +\,0 & +\,1 & x^3 - x^2 + 5x - 5 + \dfrac{6}{x+1} \\
 & +x^4 & +\,x^3 & & & & \\
 & - & - & & & & \\
\cline{2-6}
 & & -\,x^3 & +\,4x^2 & & & \\
 & & -\,x^3 & -\,x^2 & & & \\
 & & + & + & & & \\
\cline{3-6}
 & & & +\,5x^2 & +\,0 & & \\
 & & & +\,5x^2 & +\,5x & & \\
 & & & - & - & & \\
\cline{4-6}
 & & & & -\,5x & +\,1 & \\
 & & & & -\,5x & -\,5 & \\
 & & & & + & + & \\
\cline{5-6}
 & & & & & +\,6 & \text{remainder}
\end{array}
$$

In a number of special cases the quotient can be written at sight. Thus,

$$\frac{x^2 - y^2}{x - y} = x + y \qquad \frac{x^2 - y^2}{x + y} = x - y$$

$$\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2 \qquad \frac{x^3 + y^3}{x + y} = x^2 - xy + y^2$$

$$\frac{x^4 - y^4}{x - y} = x^3 + x^2y + xy^2 + y^3 \qquad \frac{x^4 - y^4}{x + y} = x^3 - x^2y + xy^2 - y^3$$

And these may be generalized as follows:

$x^n - y^n$ is always evenly divisible by $x - y$.
$x^n - y^n$ is evenly divisible by $x + y$ only when n is even.
$x^n + y^n$ is never evenly divisible by $x - y$.
$x^n + y^n$ is evenly divisible by $x + y$ only when n is odd.

Further, when $x - y$ is the divisor the signs in the quotient are all plus, but when $x + y$ is the divisor they alternate plus and minus.

18. Synthetic Division. The process of dividing a polynomial by a binomial of the form $x \pm r$ can be shortened considerably by applying synthetic division to the problem. Synthetic division is purely a mechanical procedure that eliminates all the unessential operations in the division and leaves a very simple solution. It is an abbreviated form of the ordinary long division process whereby the quotient and remainder are obtained very readily. Let us first perform a division in the ordinary way and then by synthetic division.

EXAMPLE 3–19: Divide $x^4 - 9x^3 + 26x^2 - 26x + 8$ by $x - 4$

Solution:

$$\begin{array}{l|l}
x^4 - 9x^3 + 26x^2 - 26x + 8 & x - 4^* \\ \cline{2-2}
x^4 - 4x^3 & x^3 - 5x^2 + 6x - 2 \\
- \quad + & \\ \hline
\quad - 5x^3 + 26x^2 & \\
\quad - 5x^3 + 20x^2 & \\
\quad + \qquad - & \\ \cline{1-1}
\qquad\qquad 6x^2 - 26x & \\
\qquad\qquad 6x^2 - 24x & \\
\qquad\qquad - \qquad + & \\ \cline{1-1}
\qquad\qquad\qquad - 2x + 8 & \\
\qquad\qquad\qquad - 2x + 8 & \\
\qquad\qquad\qquad + \qquad - & \\ \cline{1-1}
\end{array}$$

It is clearly evident that the work could be done just as easily by writing down only the coefficients. Thus, we would obtain:

$$\begin{array}{l|l}
1 - 9 + 26 - 26 + 8 & 1 - 4 \\ \cline{2-2}
1 - 4 & 1 - 5 + 6 - 2 \\
- \quad + & \\ \hline
\quad - 5 + 26 & \\
\quad - 5 + 20 & \\
\quad + \quad - & \\ \cline{1-1}
\qquad\quad + 6 - 26 & \\
\qquad\quad + 6 - 24 & \\
\qquad\quad - \quad + & \\ \cline{1-1}
\qquad\qquad\quad - 2 + 8 & \\
\qquad\qquad\quad - 2 + 8 & \\
\qquad\qquad\quad + \quad - & \\ \cline{1-1}
\end{array}$$

Again we can see that it is not necessary to write the first term in each line that is to be subtracted since the result in each case is zero. Therefore, the process is further shortened.

$$\begin{array}{l|l}
1 - 9 + 26 - 26 + 8 & 1 - 4 \\ \cline{2-2}
\quad - 4 & 1 - 5 + 6 - 2 \\
\quad + & \\ \cline{1-1}
\quad - 5 + 26 & \\
\qquad\quad + 20 & \\
\qquad\quad - & \\ \cline{1-1}
\qquad\quad + \ 6 - 26 & \\
\qquad\qquad\qquad - 24 & \\
\qquad\qquad\qquad + & \\ \cline{1-1}
\qquad\qquad\qquad - \quad 2 + 8 & \\
\qquad\qquad\qquad\qquad\quad + 8 & \\ \cline{1-1}
\end{array}$$

---

* The divisor is often written at the right of the dividend as indicated here.

Now, since we are omitting the first term in each line to be subtracted, we can also omit the first term in the divisor because we are no longer using it, and the total division can be condensed by writing all the subtractions on one line. Furthermore, if we copy down the first coefficient at the start of the third line, this line will be the same as the quotient, and we can omit writing the quotient. In its final form the division will then be:

$$\begin{array}{l} 1 - 9 + 26 - 26 + 8 \,\underline{|\,-4} \\ \quad - 4 + 20 - 24 + 8 \\ \underline{\quad + \quad - \quad\;\; + \quad\;\; - \quad} \\ 1 - 5 + \;\; 6 - \;\; 2 + 0 \end{array}$$

The complete procedure is easily summarized.

1. Copy down the coefficients of the dividend in the order of descending powers of x, inserting a zero where any power of x is missing.
2. Copy down the second term of the divisor at the right of the dividend.
3. Draw a line under the dividend leaving space for a row of figures above the line.
4. Bring down the first coefficient (1 in this example) under the line.
5. Multiply this coefficient by the divisor term (-4) and put the result (-4) under the second term (-9) of the dividend.
6. Subtract (-4) from (-9); change sign to $+4$ and add for a result of (-5), which makes the second term of the third line.
7. Multiply this (-5) by the divisor term (-4) and put the result $(+20)$ under the third term $(+26)$ of the dividend.
8. Subtract $(+20)$ from $(+26)$ for a result of $(+6)$ which makes the third term of the third line.
9. Multiply $(+6)$ by the divisor (-4) and put the result (-24) under the fourth term (-26) of the dividend.
10. Subtract (-24) from (-26) for a result of (-2) which makes the fourth term of the third line.
11. Multiply (-2) by the divisor (-4) and put the result $(+8)$ under the fifth and last term $(+8)$ of the dividend.
12. Subtract $(+8)$ from $(+8)$ leaving a remainder of zero in this example.

The answer can now be written directly from the third line. The last number is the remainder and the remaining numbers 1, -5, $+6$, and -2 are the coefficients of the quotient, which begins with a power of x one smaller than the dividend. Thus the answer is $x^3 - 5x^2 + 6x - 2$.

EXAMPLE 3–20. Divide $3x^3 + 5x^2 + 3x + 12$ by $x + 2$ by synthetic division.

Solution:

$$\begin{array}{l} 3 + 5 + 3 + 12 \,\underline{|\,+2} \\ \quad + 6 - 2 + 10 \\ \underline{\quad\; - \quad + \quad - \quad} \\ 3 - 1 + 5 + 2 \end{array}$$

The answer then is $3x^2 - x + 5 + \dfrac{2}{x + 2}$ where 2 is the remainder from the last subtraction.

If the divisor is not in the form $x \pm r$, both dividend and divisor should be divided through first by the coefficient of the x term in the divisor.

EXAMPLE 3-21. Divide $8x^3 - 10x^2 + 3x + 5$ by $2x + 1$.

Solution: Dividing both dividend and divisor by 2, the coefficient of the first term in the divisor, gives

$$4x^3 - 5x^2 + \frac{3}{2}x + \frac{5}{2} \quad \text{and} \quad x + \frac{1}{2}.$$

Thus we divide:

$$\begin{array}{rrrr|l} 4 & -\,5 & +\,\frac{3}{2} & +\,\frac{5}{2} & +\frac{1}{2} \\ & \pm\,2 & \mp\,\frac{7}{2} & \pm\,\frac{10}{4} & \\ \hline 4 & -\,7 & +\,\frac{10}{2} & +\,0 & \end{array}$$

The answer then is $4x^2 - 7x + 5$ with a remainder of zero.

EXERCISE 3-3

Perform the following indicated divisions:

1. $x^2 + 2x - 24$ by $x + 6$.
2. $36 + y^4 + 3y^2$ by $y^2 + 3y + 6$.
3. $a^2 + 21 - 10a$ by $a - 7$.
4. $b^4 + 16 + 4b^2$ by $4 - 2b + b^2$.
5. $25x^4 - 8 - 2x - x^2$ by $5x - 4$.
6. $a^8 + a^6 + a^4 + a^2 + 3a - 1$ by $a + 1$.
7. $y^4 + 1$ by $y - 1$.
8. $a^6 + b^6$ by $a + b$.
9. $a^6 - b^6$ by $a - b$.
10. 1 by $1 - y$ to five terms.
11. $x^4 - 3x^3 + x^2 + 2x - 1$ by $x^2 - x - 1$.
12. $a^6 - 64$ by $a - 2$.
13. $y^5 + 243$ by $y + 3$.
14. $6x^3 + 28x^4 - 6x - 2 + 6x^2$ by $2x^2 + 2x + 2$.
15. $x^7 - 2x^5 - x^3 - 10x - 36$ by $x - 2$.
16. $y^5 - 6 - 19y$ by $y + 2$.
17. $x^4 + y^4 + x^2y^2$ by $x^2 + y^2 - xy$.
18. $15 + 7y - 10y^2 - y^3 + y^4$ by $y^2 + y - 5$.
19. $12x^2 - 11xy - 36y^2$ by $3x + 4y$.
20. $2a^4 - a^3 + 4a^2 + 7a + 1$ by $a^2 + 3 - a$.
21. $y^{2n} - 28 - 3y^n$ by $y^n - 7$.
22. $21x^4 - 8x^2 - 29x^3 + 6x + 4$ by $3x - 2$.
23. $30x^4 - 36x - 62x^3 + 60x^2 + 8$ by $5x - 2$.
24. $y^6 + 12 + 38y$ by $y + 2$.
25. $2a^4 + 7a^3 - 27a^2 - 8a + 16$ by $2a^2 - 4 - 3a$.

26. $4 - 18x + 30x^2 - 23x^3 + 6x^4$ by $3x^2 + 2 - 4x$.
27. $16b - 6 + 25b^2 - 20b^3 + 3b^4$ by $b^2 - 4b - 3$.
28. $y^6 - 9y - 10 + 27y^2$ by $y^2 - 3y + 5$.
29. $a^3 + 4an^2 - 2a^2n - ax^2 - 4n^2x + 2nx^2$ by $a - x$.
30. $24x^4 - 25x - 16x^2 - 4 + 32x^3$ by $6x + 4x^2 + 1$.
31. $d^6 - 3d^5 + 6d^4 - 7d^3 + 6d^2 - 3d + 1$ by $d^2 - d + 1$.
32. $6a^3 - 13a^2b + 4ab^2 + 3b^3$ by $2a - 3b$.

19. Factoring. From the definition of a factor given in Article 2 the process of factoring is derived. It is the process of determining the two or more prime factors that when multiplied together will produce a given quantity. Thus the factors of a^2bxy are a^2, b, x, and y.

Several type forms in factoring result from the multiplication of certain quantities and occur frequently in mathematics. Therefore, it is well to consider these forms carefully so that we may be able to recognize them easily.

20. To factor a polynomial containing a common factor. A polynomial in which a common factor appears in each of its terms may be factored by dividing each term by this common factor and writing the result as the product of this common factor and the quotient.

EXAMPLE 3–22. Factor $3b^2xy - 6bx^2y + 9bxy^2$.

Solution: An examination will reveal that the factor $3bxy$ appears in each of the terms. Therefore the factors are $(3bxy)(b - 2x + 3y)$.

EXAMPLE 3–23. Factor $ax + ay + bx + by$.

Solution: The first two terms have the common factor a and the last two terms have the common factor b. Therefore, $ax + ay + bx + by = a(x + y) + b(x + y)$. Since the terms are alike, their coefficients can be added and we have $a(x + y) + b(x + y) = (a + b)(x + y)$.

21. To factor a trinomial that is a perfect square. In multiplication, we have found that

$$(x + y)^2 = x^2 + 2xy + y^2.$$

and

$$(x - y)^2 = x^2 - 2xy + y^2.$$

An examination of these perfect square forms shows that two of the terms are perfect squares, x^2 and y^2, and the middle term is equal numerically to two times the product of the square roots of the squared terms. When these conditions are fulfilled, the expression is a perfect square and can be easily factored into its two equal roots, each root being found by taking the square root of each squared term and connecting them with the sign of the middle term.

Thus,
$$a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2$$
and
$$a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2.$$

Sometimes it will not be apparent that the expression is a perfect square until a common term has also been factored out.

EXAMPLE 3–24. Factor $x^3 + 2x^2y + xy^2$.

Solution: $x^3 + 2x^2y + xy^2 = x(x^2 + 2xy + y^2) = (x)(x + y)(x + y)$.

22. To factor the difference of two squares. We have seen that in multiplication $(x + y)(x - y) = x^2 - y^2$. Therefore, to factor the difference of two squares, find the square roots of the two terms and make their sum one factor and their difference the other factor.

EXAMPLE 3–25. Factor $4x^2 - 9y^2$.

Solution: $4x^2 - 9y^2 = (2x)^2 - (3y)^2 = (2x + 3y)(2x - 3y)$.

EXAMPLE 3–26. Factor $16a^2 - (2a + b)^2$.

Solution:

$$\begin{aligned} 16a^2 - (2a + b)^2 &= (4a)^2 - (2a + b)^2 \\ &= [4a + (2a + b)][4a - (2a + b)] \\ &= [4a + 2a + b][4a - 2a - b] \\ &= (6a + b)(2a - b). \end{aligned}$$

EXAMPLE 3–27. Factor $x^2 - a^2 - 2xy + y^2 + 2a - 1$.

Solution:

$$x^2 - a^2 - 2xy + y^2 + 2a - 1 = (x^2 - 2xy + y^2) - (a^2 - 2a + 1).$$

(Note the change in signs of all terms in the second polynomial due to being placed in negative parentheses.)

$$\begin{aligned} (x^2 - 2xy + y^2) - (a^2 - 2a + 1) &= (x - y)^2 - (a - 1)^2 \\ &= [(x - y) + (a - 1)][(x - y) - (a - 1)] \\ &= (x - y + a - 1)(x - y - a + 1). \end{aligned}$$

The principle of factoring the difference of two squares has some special applications well worth consideration.

EXAMPLE 3–28. Factor $x^4 + x^2y^2 + y^4$.

Solution: Now, since $x^4 + x^2y^2 + y^4$ lacks only an x^2y^2 of being a perfect square, a method of factoring readily suggests itself. By adding the x^2y^2 and also subtracting it, the value of the original expression is not changed but the expression is converted into the difference of two squares. Thus by adding and subtracting x^2y^2 there results $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2 + y^2)^2 - x^2y^2$. The factors of this last expression may now be easily recognized: $(x^2 + y^2)^2 - x^2y^2 = (x^2 + y^2 + xy)(x^2 + y^2 - xy)$.

EXAMPLE 3–29. Factor $4x^4 + 8x^2y^2 + 9y^4$.

Solution:

$$\begin{aligned} 4x^4 + 8x^2y^2 + 9y^4 &= 4x^4 + 12x^2y^2 + 9y^4 - 4x^2y^2, \\ 4x^4 + 12x^2y^2 + 9y^4 - 4x^2y^2 &= (2x^2 + 3y^2)^2 - 4x^2y^2, \\ (2x^2 + 3y^2)^2 - 4x^2y^2 &= (2x^2 + 3y^2 + 2xy), \\ &\quad\ (2x^2 + 3y^2 - 2xy). \end{aligned}$$

EXAMPLE 3–30. Factor $a^4 + 2a^2b^2 - 3b^4$.

Solution: If the $-3b^4$ were $+b^4$, then we would have a perfect square. By adding $+4b^4$ and also subtracting $+4b^4$ we make a perfect square of this expression and at the same time get the difference of two squares to factor.

Thus, $a^4 + 2a^2b^2 - 3b^4$ becomes

$$a^4 + 2a^2b^2 - 3b^4 + 4b^4 - 4b^4.$$

So
$$\begin{aligned} a^4 + 2a^2b^2 - 3b^4 &= a^4 + 2a^2b^2 + b^4 - 4b^4 \\ &= (a^2 + b^2)^2 - 4b^4 \\ &= (a^2 + b^2 + 2b^2)(a^2 + b^2 - 2b^2) \\ &= (a^2 + 3b^2)(a^2 - b^2). \end{aligned}$$

23. To factor a trinomial of the type form $x^2 + bx + c$. In multiplication,

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

and
$$(x + 2)(x + 3) = x^2 + (2 + 3)x + 6.$$

Now by reversing the process a trinomial of this form may be factored by finding two factors of the absolute term (c) such that their sum is equal to b, the coefficient of the x term, and then adding algebraically each factor to x. Thus,

$$\begin{aligned} x^2 + 9x + 14 &= (x + 7)(x + 2), \\ x^2 - 9x + 14 &= (x - 7)(x - 2), \\ x^2 + 5x - 14 &= (x + 7)(x - 2), \\ x^2 - 5x - 14 &= (x - 7)(x + 2). \end{aligned}$$

EXAMPLE 3–31. Factor $a^2 + 7a + 12$.

Solution: The two factors of $+12$ that give $+7$ when added together are $+3$ and $+4$. Therefore,

$$a^2 + 7a + 12 = (a + 3)(a + 4).$$

If the expression had been $a^2 - 7a + 12$, the factors would have been $(a - 3)(a - 4)$.

EXAMPLE 3–32. Factor $a^2 + a - 12$.

Solution: The two factors of -12 that give $+1$ when added are $+4$ and -3. Therefore, $a^2 + a - 12 = (a + 4)(a - 3)$. If the expression had been $a^2 - a - 12$, the factors would have been $(a - 4)(a + 3)$.

24. To factor a trinomial of the type form $ax^2 + bx + c$. This form is similar to the type form $x^2 + bx + c$ and the same method is applied. Factors for the first term, ax^2, and for the third term, c, are found. Then a trial process is used to determine which of these factors will give the middle term.

EXAMPLE 3–33. Factor $10x^2 + 29x + 12$.

Solution: The factors of $10x^2$ are: $10x$ and x,
$5x$ and $2x$.
The factors of 12 are: 12 and 1,
6 and 2,
4 and 3.

Arranging these factors in all the possible combinations gives the following:

1. $(10x + 3)(x + 4)$.
2. $(10x + 4)(x + 3)$.
3. $(10x + 2)(x + 6)$.
4. $(10x + 6)(x + 2)$.
5. $(10x + 1)(x + 12)$.
6. $(10x + 12)(x + 1)$.
7. $(5x + 3)(2x + 4)$.
8. $(5x + 4)(2x + 3)$.
9. $(5x + 2)(2x + 6)$.
10. $(5x + 6)(2x + 2)$.
11. $(5x + 12)(2x + 1)$.
12. $(5x + 1)(2x + 12)$.

All that is required is to select the combination that will result in $+29x$ for the second term. An examination of these twelve will show that combination No. *11* is the only one that will produce $29x$ in the middle term. Therefore, the factors of $10x^2 + 29x + 12$ are $(5x + 12)(2x + 1)$.

It is possible that more than one combination might be found that would satisfy the requirements for the middle term, and in such a case there would be as many sets of factors as there are combinations that satisfy the given conditions. Thus, in these twelve combinations, $(10x + 4)(x + 3)$ and $(5x + 2)(2x + 6)$ both result in a middle term of $34x$. Therefore, the factors of $10x^2 + 34x + 12$ would be

$$(10x + 4)(x + 3)$$

or

$$(5x + 2)(2x + 6).$$

EXAMPLE 3–34. Factor $8x^2 - 14x - 15$.

Solution: The factors of $8x^2$ are: $8x$ and x,
$4x$ and $2x$.
The factors of 15 are: 15 and 1,
5 and 3.

Now, since the last term is -15, one of its factors must be plus and the other minus. Also since the middle term is minus, the product of the minus factor of -15 and one of the factors of $8x^2$ must be larger than the product of the plus factor of -15 and the other factor of $8x^2$. The possible combinations are:

1. $(8x + 15)(x - 1)$.
2. $(8x - 15)(x + 1)$.
3. $(8x + 1)(x - 15)$.
4. $(8x - 1)(x + 15)$.
5. $(8x + 5)(x - 3)$.
6. $(8x - 5)(x + 3)$.
7. $(8x + 3)(x - 5)$.
8. $(8x - 3)(x + 5)$.
9. $(4x + 15)(2x - 1)$.
10. $(4x - 15)(2x + 1)$.
11. $(4x + 1)(2x - 15)$.
12. $(4x - 1)(2x + 15)$.
13. $(4x + 5)(2x - 3)$.
14. $(4x - 5)(2x + 3)$.
15. $(4x + 3)(2x - 5)$.
16. $(4x - 3)(2x + 5)$.

The only combination here that will satisfy all the conditions and result in $-14x$ for the middle term is No. *15*, because the algebraic sum of $(4x)(-5)$ and $(3)(2x)$ is $-14x$.

25. Special cases in factoring. In division several special cases were enumerated; and these cases apply equally well in factoring. Thus,

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3) = (x - y)(x + y)(x^2 + y^2)$$
$$x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$$
$$x^4 - y^4 = (x + y)(x^3 - x^2y + xy^2 - y^3) = (x + y)(x - y)(x^2 + y^2)$$
$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$$
$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$$

Note that in the case of the difference of the same two even powers, as $x^4 - y^4$, there are several ways of factoring, though the final results come out with the same prime factors.

26. The factor theorem. If none of the above methods of factoring seem to fit, then the factor theorem may be employed. This theorem, briefly stated, says that a polynomial is exactly divisible by $x - (\pm r)$, if by substituting r for x the polynomial reduces to zero.

EXAMPLE 3–35. Factor $x^3 + 2x^2 - x - 2$.

Solution: By substituting a value of 1 for x in this expression there results $(1)^3 + (2)(1)^2 - (1) - 2 = 1 + 2 - 1 - 2 = 0$. Since the substitution of 1 for x makes $x^3 + 2x^2 - x - 2$ equal to 0, then one factor will be $x - 1$. Dividing $x^3 + 2x^2 - x - 2$ by $x - 1$ gives $x^2 + 3x + 2$, which may be factored at sight.

$$\begin{aligned} x^3 + 2x^2 - x - 2 &= (x - 1)(x^2 + 3x + 2) \\ &= (x - 1)(x + 1)(x + 2). \end{aligned}$$

This expression might have been solved by the substitution of -1 for x in the original polynomial. The result would then be $(-1)^3 + 2(-1)^2 - (-1) - 2 = -1 + 2 + 1 - 2 = 0$. Since the substitution of -1 also makes the polynomial equal to zero, then one factor is $x - (-1)$ or $x + 1$. Dividing $x^3 + 2x^2 - x - 2$ by $x + 1$ gives $x^2 + x - 2$, which can be factored into $(x + 2)(x - 1)$. Therefore, the three factors are $(x + 1)(x + 2)(x - 1)$, the same as before.

Again, the expression might have been solved by the substitution of -2 for x in the original polynomial. The result would then be $(-2)^3 + 2(-2)^2 - (-2) - 2 = -8 + 8 + 2 - 2 = 0$. Since the substitution of -2 makes the polynomial equal to zero, then one factor is $x - (-2)$ or $x + 2$. Dividing $x^3 + 2x^2 - x - 2$ by $x + 2$ gives $x^2 - 1$, which can be factored into $(x + 1)(x - 1)$. Therefore, the three factors are $(x + 2)(x + 1)(x - 1)$, the same result as in the other cases.

It should be noted that the various values substituted for x were all factors of the constant -2, at the right in the polynomial.

EXERCISE 3-4

Factor the following polynomials:

1. $6ax^2y - 12a^2xy^2 + 24a^3xy$.
2. $5x^4y - 20x^3y^2 + 15xy^3$.
3. $x^9 + x^8 - x^7 + x^6 - x^5$.
4. $6a^3b^3 - 3a^2b^2 + 9ab$.
5. $24r^2s^2 - 48r^2s^4 + 96r^3s^3$.
6. $ax - ay - bx + by$.
7. $y^3 - y^2 - 2y + 2$.
8. $14ax + 10ay - 7bx - 5by$.
9. $ay + a - by - b - cy - c$.
10. $15a^3 - 12ab - 5a^4 + 4a^2b$.
11. $x^2 + 12x + 36$.
12. $3x^2 - 18x + 27$.
13. $1 + 8ab + 16a^2b^2$.
14. $16(x + y)^2 + 40(x + y) + 25$.
15. $25a^4 + 10a^2b^2 + b^4$.
16. $256 - x^4$.
17. $81y^4 - 1$.
18. $b^2 - y^2$.
19. $900x^2 - 169y^2$.
20. $(3b + 2a)^2 - (y - 2x)^2$.
21. $a^{2x} - y^{2x}$.
22. $36x^2 - (5x + 2y)^2$.
23. $16a^2 - (b - a)^2$.
24. $a^2 - 2ab + b^2 - x^2 + 12x - 36$.
25. $35 - 2x - x^2$.
26. $x^6 + 11x^3a - 42a^2$.
27. $x^2 + 12x + 35$.
28. $2x^2 - 27ax - 45a^2$.
29. $3a^2 - 24a - 315$.
30. $2bx + b^2 - 48x^2$.
31. $4x^4 + 72x^2y^2 + 288y^4$.
32. $x^8 + 10x^4y + 21y^2$.
33. $2x^4 - 26b^3x^2 - 60b^6$.
34. $x^4 + 7x^2y^2 - 144y^4$.
35. $12a^2 - 7a - 45$.
36. $8x^2 - 17x - 21$.
37. $4a^2 - 17ab + 4b^2$.
38. $21b^2 + 50b - 16$.
39. $20a^2 + 7a - 3$.
40. $14x^2 + 12x - 32$.
41. $10x^2 + x - 21$.
42. $16a^2 - 32a - 20$.
43. $14y^6 - 17y^3 - 6$.
44. $24y^4 + 30y^2 - 99$.
45. $(x + y)^3 - 8$.
46. $a^3 + 1$.
47. $2x^5 - 64$.
48. $x^8 - y^8$.
49. $x^4 + 4$.
50. $a^3 - 9a^2 + 23a - 15$.
51. $x^3 + 7x^2 + 2x - 40$.
52. $by^3 - 3by^2 + 26by - 24b$.
53. $a^3b - 15a^2b + 38ab - 24b$.
54. $a^3 - 11a^2 + 31a - 21$.
55. $(x - 1)^3 + (x - 2)^3$.
56. $x^3 - 5x^2 - 29x + 105$.
57. $ay^2 - 11ay - 42a$.
58. $2amx - 2bnx + 3am - 3bn$.
59. $a^3 - 16a^2 + 71a - 56$.
60. $a^2b^2 + b^2x^2 + x^2y^2 - 2ab^2x + 2abxy - 2bx^2y$.
61. $(27.3)^2 - (2.7)^2$.
62. $(137.4)^2 - (19.6)^2$.
63. $16a^4 - 28a^2b^2 + 9b^4$.
64. $1 + 3ab - 10a^2b^2$.
65. $(a + b)^5 - c^5$.
66. $\frac{1}{9}x^2 + \frac{1}{3}xy + \frac{1}{4}y^2$.
67. $t^2 - r^2 + s^2 - 2st$.
68. $x^{2m} + 2 + \frac{1}{x^{2m}}$.
69. $3x^2 + 9xy - \frac{10}{3}y^2$.

70. Show that $x + 2$ is a factor of $x^5 + 4x^2 + 16$ without performing the division.

71. Is $x - 2$ also a factor of $x^5 + 4x^2 + 16$? Why?

72. Find the three factors of $x^3 + 2x^2 - 5x - 6$ without performing any divisions.

27. Fractions. A fraction in algebra has exactly the same meaning as a fraction in arithmetic and the same terms and principles are applied. Fractions with the same denominator are *similar fractions* while those with different denominators are *dissimilar fractions.*

The sign written in front of the fraction belongs to the fraction as a whole rather than to either the numerator or denominator. However, the numerator and denominator will each have a sign of its own. In the fraction $-\frac{x}{2y}$ the sign of the fraction is minus, while the sign of the numerator is plus and the sign of the denominator is also plus. In the fraction, $-\frac{-2x}{y}$, the sign of the fraction is minus, the sign of the numerator is minus, and the sign of the denominator is plus. It should be noted that when no sign is indicated a positive sign is understood. Both the sign of the numerator and the sign of the denominator can be changed without affecting the value of the fraction. But if only one sign is changed (either the numerator or the denominator), the sign of the fraction must also be changed in order to keep the same value for the fraction. Thus,

$$+\frac{2x}{3y} = +\frac{-2x}{-3y} = -\frac{-2x}{3y} = -\frac{2x}{-3y}.$$

The following rules are obtained for the signs in fractions:

1. *If the signs of both the numerator and the denominator are changed, the sign of the fraction remains unchanged.*

2. *If the sign of either the numerator or the denominator (but not both) is changed, the sign of the fraction must be changed.*

If the numerator or denominator is a polynomial, then, when changing the sign of either, the sign of each term therein must be changed. Thus,

$$+\frac{a+b}{x-y} = +\frac{-(a+b)}{-(x-y)} = +\frac{-a-b}{-x+y},$$

$$+\frac{a+b}{x-y} = -\frac{-(a+b)}{x-y} = \frac{-a-b}{x-y},$$

$$+\frac{a+b}{x-y} = -\frac{a+b}{-(x-y)} = -\frac{a+b}{-x+y}.$$

It is important to note that the sign of each term making up the numerator or denominator must be changed when the sign of either the numerator or the denominator is to be changed. Thus,

$$-\frac{a+b}{x-y} = +\frac{a+b}{-(x-y)} = +\frac{a+b}{-x+y}.$$

But $$-\frac{a+b}{x-y} \neq -\frac{a+b}{-x-y},$$

because the sign of each term in the denominator has not been changed. (*Note:* The operational sign $\neq$ means "is not equal to.") Now from the law of multiplication, it is known that the product of two negative factors is positive; of three negative factors, negative; of four negative factors, positive; of five negative factors, negative; and so on. Hence, it is apparent that

3. The sign of either term of a fraction is not affected by changing the sign of an even number of its factors.

4. The sign of either term of a fraction is changed by changing the signs of an odd number of its factors.

28. Operations on the numerator and denominator of fractions. When fractions are to be affected by one of the four fundamental operations, it becomes necessary to make use of the following principles:

1. The value of a fraction remains unchanged when the numerator and denominator are multiplied by the same number or expression, other than zero.

2. The value of a fraction remains unchanged when the numerator and denominator are divided by the same number or expression, other than zero.

3. The value of a fraction is changed when the numerator or denominator or both are changed by the addition or subtraction of some number or quantity, other than zero.

Thus,
$$\frac{2}{3} = \left(\frac{2}{3}\right)\left(\frac{2}{2}\right) = \frac{4}{6},$$

and
$$\frac{4}{8} = \frac{\frac{4}{4}}{\frac{8}{4}} = \frac{1}{2}.$$

But
$$\frac{4}{5} \neq \frac{4+2}{5+2}, \text{ because } \frac{4+2}{5+2} = \frac{6}{7},$$

and
$$\frac{4}{5} \neq \frac{4-2}{5-2}, \text{ because } \frac{4-2}{5-2} = \frac{2}{3}.$$

We have not evolved any new principles here, because multiplying or dividing the numerator and denominator by the same number, except zero, is equivalent to multiplying or dividing the fraction by 1.

It should be noted that the multiplication or division by zero is excluded from these principles. For example, $5 \times 0 = 0$, and therefore the multiplication of both numerator and denominator by zero would result in $\frac{0}{0}$, which is an indeterminate form.

Division by zero has no meaning whatsoever, because it results in an infinitely large value. As proof of this, consider the division of 1 by smaller and smaller denominators.

Thus,

$$\frac{1}{1} = 1 \qquad \frac{1}{\frac{1}{100{,}000}} = 100{,}000$$

$$\frac{1}{\frac{1}{10}} = 10 \qquad \frac{1}{\frac{1}{10{,}000{,}000}} = 10{,}000{,}000$$

$$\frac{1}{\frac{1}{1{,}000}} = 1{,}000$$

It is evident that the answer is getting larger and larger as the denominator becomes smaller and smaller, and, if the denominator is taken to its smallest possible value of zero, the answer will become infinitely large and not computable.

29. Reduction of fractions. Reduction is the process of changing the form of an expression without changing its value. This process may be: changing a fraction to lower or higher terms; changing a fraction to an integral expression or to a mixed expression; changing integral or mixed expressions to fractional form; changing two or more fractions to a common denominator.

The value of a fraction will not be changed if both numerator and denominator are multiplied by the same quantity or divided by the same quantity. Thus,

$$\frac{x}{y} = \frac{ax}{ay} \quad \text{or} \quad \frac{ax}{ay} = \frac{x}{y}.$$

It is often desirable to reduce a fraction to its lowest terms. To do this the numerator and denominator are factored and like terms canceled out.

EXAMPLE 3–36. Reduce $\dfrac{x^2 - y^2}{x^2 - 2xy + y^2}$ to its lowest terms.

Solution:

$$\frac{x^2 - y^2}{x^2 - 2xy + y^2} = \frac{(x + y)\cancel{(x - y)}}{(x - y)\cancel{(x - y)}} = \frac{x + y}{x - y}.$$

It is very important that the student understand clearly what may be canceled in the numerator and denominator. For instance, in the solution to Example 3–36, $\dfrac{x + y}{x - y}$, it is not correct to cancel out the x's and leave $\dfrac{+ y}{- y}$, or -1. The numerator, $x + y$, represents one single quantity and the denominator, $x - y$, represents one single quantity, and while it is permissible to cancel whole quantities it is not permissible to cancel parts of quantities. The difference is shown in the following:

$$\frac{(x + y)(x - y) + 2}{x - y} \text{ is not equal to } \frac{(x + y)\cancel{(x - y)} + 2}{\cancel{x - y}}$$

because $(x - y)$ in the numerator is only a part of a quantity. If the expression were written as

$$\frac{(x + y)(x - y) + 2(x - y)}{x - y},$$

then it would be permissible to cancel $(x - y)$ because the expression could be changed to read

$$\frac{(x - y)(x + y + 2)}{x - y},$$

in which the $(x - y)$ in the numerator is a common factor, and represents a whole quantity. In general, it may be said that cancellation is permissible only where there are multiplication and division of factors. If there is addition or subtraction of terms, then common factors must be found before cancellation is permitted. Therefore, common or *like factors* may be canceled from the numerator and denominator but *like terms* may not be canceled.

Again, it may be necessary to change a fraction to one having a larger denominator.

EXAMPLE 3–37. Change $\frac{a - b}{a + b}$ to a fraction whose denominator is $a^2 - b^2$.

Solution: Since the new denominator is to be $a^2 - b^2$, the old denominator, $a + b$, must be multiplied by $a - b$ to give $a^2 - b^2$, and to keep the value of the fraction the same the numerator also must be multiplied by $a - b$.

$$\frac{(a - b)(a - b)}{(a + b)(a - b)} = \frac{a^2 - 2ab + b^2}{a^2 - b^2}.$$

The reduction of a fraction to an integral expression or to a mixed expression is accomplished by performing as much of the indicated division as is possible.

EXAMPLE 3–38. Reduce $\frac{ax + by}{b}$ to a mixed expression.

Solution:

$$\frac{ax + by}{b} = \frac{ax}{b} + \frac{by}{b} = \frac{ax}{b} + y.$$

EXAMPLE 3–39. Reduce $\frac{x^3 - 8x^2 + 19x - 12}{x^2 - 4x + 3}$ to an integral expression.

Solution: Factor both numerator and denominator first.

$$\frac{x^3 - 8x^2 + 19x - 12}{x^2 - 4x + 3} = \frac{(x - 4)\cancel{(x - 3)}\cancel{(x - 1)}}{\cancel{(x - 3)}\cancel{(x - 1)}} = x - 4.$$

30. Addition and subtraction of fractions. It is necessary to change fractions to the same denominator in order to add or subtract them. This denominator is generally the least common denominator, which is defined as the smallest quantity into which each and every denominator will divide exactly. To find the least common denominator, the following steps are taken:

1. Determine the smallest quantity that will exactly contain all the denominators.

2. Divide this quantity by the denominator of the first fraction and then multiply this result by the numerator and denominator of the fraction to produce a new equal fraction.

3. Repeat for all other fractions.

EXAMPLE 3–40. Reduce $\frac{a}{3}$ and $\frac{b}{4x}$ to a common denominator.

Solution: The least common denominator is $12x$. Therefore,

$$\frac{a}{3} = \frac{(a)(4x)}{(3)(4x)} = \frac{4ax}{12x}$$

and

$$\frac{b}{4x} = \frac{(b)(3)}{(4x)(3)} = \frac{3b}{12x}.$$

EXAMPLE 3–41. Reduce $\frac{4x}{x - y}$ and $\frac{x - y}{x + y}$ to a common denominator.

Solution: The least common denominator is $(x - y)(x + y)$ or $x^2 - y^2$. Therefore,

$$\frac{4x}{x - y} = \frac{(4x)(x + y)}{(x - y)(x + y)} = \frac{4x^2 + 4xy}{x^2 - y^2},$$

$$\frac{x - y}{x + y} = \frac{(x - y)(x - y)}{(x + y)(x - y)} = \frac{x^2 - 2xy + y^2}{x^2 - y^2}.$$

After reducing two or more fractions to their least common denominator, the sum can be found by adding all the numerators together and placing this result over the common denominator. If it is required to subtract one fraction from another, the signs of the terms to be subtracted are reversed and the numerators are then added and the result placed over the common denominator.

EXAMPLE 3–42. Add the following: $\frac{3x}{5}, \frac{4y}{15}$, and $\frac{7x}{10}$.

Solution: $\frac{3x}{5} + \frac{4y}{15} + \frac{7x}{10} = \frac{18x}{30} + \frac{8y}{30} + \frac{21x}{30} = \frac{39x + 8y}{30}.$

EXAMPLE 3–43. Simplify the following by changing all fractions to a common denominator and adding or subtracting as indicated:

$$\frac{4x}{x + y} - 3 + \frac{8xy}{x^2 - y^2} - \frac{5y}{x - y}.$$

Solution: Since no denominator is indicated under the second term, $- 3$, its denominator is understood to be 1, and the least common denominator for the four terms is $x^2 - y^2$. Therefore

$$\begin{aligned}\frac{4x}{x+y} - \frac{3}{1} + \frac{8xy}{x^2-y^2} - \frac{5y}{x-y} &= \frac{4x(x-y)}{x^2-y^2} - \frac{3(x^2 - y^2)}{x^2-y^2} + \frac{8xy}{x^2-y^2} - \frac{5y(x+y)}{x^2-y^2} \\ &= \frac{4x^2-4xy}{x^2-y^2} - \frac{3x^2-3y^2}{x^2-y^2} + \frac{8xy}{x^2-y^2} - \frac{5xy+5y^2}{x^2-y^2} \\ &= \frac{4x^2-4xy-(3x^2-3y^2)+8xy-(5xy+5y^2)}{x^2 - y^2}\end{aligned}$$

$$= \frac{4x^2 - 4xy - 3x^2 + 3y^2 + 8xy - 5xy - 5y^2}{x^2 - y^2}$$
$$= \frac{x^2 - xy - 2y^2}{x^2 - y^2} = \frac{(x - 2y)(x + y)}{x^2 - y^2}$$
$$= \frac{(x - 2y)(x + y)}{(x + y)(x - y)} = \frac{x - 2y}{x - y}.$$

31. Multiplication and division of fractions. Multiplication of fractions in algebra is exactly the same as in arithmetic. Thus,

$$\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \frac{8}{15}$$

or, in general,

$$\left(\frac{a}{b}\right)\left(\frac{x}{y}\right) = \frac{ax}{by}.$$

Thus, the product of any number of fractions is equal to the product of their numerators divided by the product of their denominators. Similar factors may be canceled out to simplify the result. Dividing by a fraction is the same as multiplying by the reciprocal of the fraction. Like factors in numerator and denominator may be canceled out, as in multiplication.

EXAMPLE 3–44. Multiply $\frac{x + 2}{x - 2}$ by $x^2 - 4$.

Solution:

$$\left(\frac{x + 2}{\cancel{x - 2}}\right)\overset{x+2}{\cancel{(x^2 - 4)}} = (x + 2)(x + 2) = x^2 + 4x + 4.$$

EXAMPLE 3–45. Multiply $\frac{x + 1}{x + 2}$ by $\frac{1}{x + 2} + 2$.

Solution:

$$\left(\frac{x + 1}{x + 2}\right)\left(\frac{1}{x + 2} + \frac{2}{1}\right) = \left[\frac{x + 1}{x + 2}\right]\left[\frac{(1)}{x + 2} + \frac{(2)(x + 2)}{(1)(x + 2)}\right]$$
$$= \frac{(x + 1)(1 + 2x + 4)}{(x + 2)(x + 2)} = \frac{(x + 1)(2x + 5)}{(x + 2)(x + 2)}$$
$$= \frac{2x^2 + 7x + 5}{x^2 + 4x + 4}.$$

EXAMPLE 3–46. Simplify: $\dfrac{\left(\dfrac{x^2 + 2x - 15}{x^2 - 16}\right)\left(\dfrac{x^2 + 4x}{x^2 + 3x - 10}\right)}{\dfrac{x^2 - 3x}{x^2 + x - 20}}$.

Solution:

$$\frac{\left(\dfrac{x^2 + 2x - 15}{x^2 - 16}\right)\left(\dfrac{x^2 + 4x}{x^2 + 3x - 10}\right)}{\dfrac{x^2 - 3x}{x^2 + x - 20}} = \frac{\left[\dfrac{(x + 5)(x - 3)}{(x + 4)(x - 4)}\right]\left[\dfrac{x(x + 4)}{(x + 5)(x - 2)}\right]}{\dfrac{(x)(x - 3)}{(x + 5)(x - 4)}}$$
$$= \left[\frac{(x + 5)(x - 3)(x)(x + 4)}{(x + 4)(x - 4)(x + 5)(x - 2)}\right]\left[\frac{(x + 5)(x - 4)}{(x)(x - 3)}\right]$$
$$= \frac{\cancel{(x + 5)}\cancel{(x - 3)}\cancel{(x)}\cancel{(x + 4)}(x + 5)\cancel{(x - 4)}}{\cancel{(x + 4)}\cancel{(x - 4)}\cancel{(x + 5)}(x - 2)\cancel{(x)}\cancel{(x - 3)}} = \frac{x + 5}{x - 2}.$$

EXAMPLE 3–47. Simplify $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{a}}}$.

Solution: It will be necessary to start simplifying at the bottom with the part $\dfrac{1}{1+\dfrac{1}{a}}$.

Thus, $$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{a}}} = \frac{1}{1+\dfrac{1}{\dfrac{a+1}{a}}} = \frac{1}{1+\dfrac{a}{a+1}} = \frac{1}{\dfrac{a+1+a}{a+1}} = \frac{a+1}{2a+1}.$$

EXERCISE 3–5

Simplify each of the following problems:

1. $\dfrac{18ab^2xy^3}{12a^2bx^2y}$.

2. $\dfrac{2m^2st^3}{3mnst}$.

3. $-\dfrac{a^2-b^2}{b-a}$.

4. $\dfrac{12x^3-18x^2y^2}{4x-6xy}$.

5. $-\dfrac{2x-1}{1-4x^2}$.

6. $\dfrac{\dfrac{a^2-b^2}{xy}}{\dfrac{a+b}{xy^2}}$.

7. $\dfrac{3-4a}{a}-\dfrac{a-5}{1-2a}-\dfrac{5a^2+1}{a-2a^2}$.

8. $\dfrac{V^2-10V+21}{V^2+2V-15}$.

9. $\dfrac{\dfrac{a+b}{b}-\dfrac{a+b}{a}}{\dfrac{1}{b}-\dfrac{1}{a}}$.

10. $\dfrac{x^4+x^2+1}{x^3+x^2+x}$.

11. $\dfrac{(x+y)^2-z^2}{(x+z)^2-y^2}$.

12. $\dfrac{6a-\dfrac{1}{a}-1}{\dfrac{2a-1}{3a}}$.

13. $\dfrac{x^2-4x-5}{x^2-5x}$.

14. $\dfrac{\dfrac{x^4}{y^4}+\dfrac{2x^2}{y^2}-3}{\dfrac{x^4}{y^4}+\dfrac{x^2}{y^2}+1}$.

15. $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-R}}}}$.

16. $\dfrac{1}{R+\dfrac{1}{\dfrac{1}{R}+\dfrac{1}{x}}}$.

17. $\dfrac{x}{x+1+\dfrac{x}{x+1-\dfrac{1}{x}}}$.

18. $\dfrac{1}{4}+\dfrac{x}{x-y}-\dfrac{2x^2}{y^2-x^2}-\dfrac{x}{x+y}$.

19. $\dfrac{\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}}{\dfrac{x^3+y^3}{x^2-y^2}}$.

20. $\dfrac{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}{\dfrac{x+y}{2(x-y)}-\dfrac{x-y}{2(x+y)}}$.

21. $\dfrac{\dfrac{\dfrac{1}{x}+\dfrac{1}{y+z}}{\dfrac{1}{x}-\dfrac{1}{y+z}}}{\dfrac{1}{1+\dfrac{y^2+z^2-x^2}{2yz}}}$.

22. $\dfrac{\dfrac{1}{y+1}}{1-\dfrac{1}{y+1}}+\dfrac{\dfrac{y+1}{y}}{\dfrac{y}{1-y}}+\dfrac{\dfrac{1}{1-y}}{\dfrac{y}{y+1}}$.

23. $\dfrac{R_1-\dfrac{R_1R_2}{R_1+R_2}}{\dfrac{R_1R_2}{R_1-R_2}+R_1}$.

24. $\dfrac{1}{1+\dfrac{x-1}{1-\dfrac{x}{3}-\dfrac{1}{x}}}$.

25. $\dfrac{\dfrac{x+y}{x^2-y^2}-\dfrac{1}{x+y}}{\dfrac{1}{x+y}-\dfrac{x}{x^2-y^2}}$.

26. $\dfrac{1-2x}{4x^2-1}$.

27. $\dfrac{a^2b-ab}{1-a^2}$.

28. $\dfrac{b^{2y}-a^{2x}}{b^y+a^x}$.

29. $\dfrac{x}{x-y}+\dfrac{y}{x+y}+\dfrac{x^2+y^2}{y^2+x^2}$.

30. $\dfrac{4x-3}{x}-\dfrac{1-x}{2x-1}+\dfrac{5x^2+1}{x-2x^2}$.

31. $\left[\dfrac{-a-b}{b-a}\right]\left[\dfrac{a^2-b^2}{(a+b)^2}\right]$.

32. $\left[\dfrac{5(3-x)}{-2(x+1)}\right]\left[\dfrac{4(x+1)(x+2)}{15(x-3)}\right]$.

33. $\dfrac{\dfrac{-8a^3}{b^3-a^3}}{\dfrac{4a^3}{a^2+ab+b^2}}$.

34. $\dfrac{\dfrac{x^3+3x^2y+3xy^2+y^3}{a^3+b^3}}{\dfrac{(x+y)^2}{a^2-ab+b^2}}$.

35. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

36. $\dfrac{\left[\dfrac{x^2+y^2}{y}-x\right]\left[\dfrac{x^2-y^2}{x^3+y^3}\right]}{\dfrac{1}{y}-\dfrac{1}{x}}$.

37. $\dfrac{\dfrac{(a+2)(a-2)^2}{(a+3)(a^3-27)}}{\dfrac{(a+7)(a+2)}{a^2+6a+9}}$
$\overline{\dfrac{9-a^2}{4a-a^2-4}}$.

38. $\dfrac{1}{2}+\dfrac{b}{a-b}+\dfrac{2a^2}{b^2-a^2}+\dfrac{-a}{a+b}$.

39. $\dfrac{x^2+4xy}{x^3+y^3}+\dfrac{1}{x+y}-\dfrac{x}{x^2-xy+y^2}$.

40. $\dfrac{\dfrac{x^2+9xy+18y^2}{x^2-9xy+20y^2}}{\dfrac{x^2+6xy+9y^2}{xy^2-4y^3}}$.

41. $\dfrac{\dfrac{x-3}{x+4}}{\dfrac{x+2}{x-4}}$
$\overline{\dfrac{x-4}{x+2}}$.

42. $\dfrac{\left(\dfrac{b^2-a^2}{ab^2x}\right)\left(\dfrac{b(a-b)}{a^2+2ab+b^2}\right)}{\dfrac{a^2-2ab+b^2}{b(a+b)}}$.

43. $\left[\dfrac{12c^3b}{5(c^3-b^3)}\right]\left[\dfrac{35(c^2+cb+b^2)}{14c^2b^2}\right]$.

44. $\dfrac{(3t^2-2t-1)(2t^2+5t-3)(4t^2+10t+4)}{(2t^2+t-1)(3t^2+7t+2)(4t^2-2t-2)}$.

45. $\dfrac{a^2+2a-15}{a^2-5a+6}-\dfrac{5a-4}{2-a}$.

46. $\dfrac{12xy}{9x^2-4y^2}+\dfrac{x+y}{2y-3x}+\dfrac{x-y}{3x+2y}$.

47. $\left(\frac{16 - x^4}{(x^2 - 4)^2}\right)\left(\frac{4 - x^2}{x^2 + 4}\right)$.

48. $\left(\frac{2a^2-5ax-3x^2}{9a^2 - x^2}\right)\left(\frac{-3x-9a}{10a^2+5ax}\right)$.

49. $\dfrac{\frac{a^2 - 4x^2}{a^2 + 4ax}}{\frac{a^2 - 2ax}{ax + 4x^2}}$.

50. $\dfrac{a + b + \frac{b^2}{a}}{a + b + \frac{a^2}{b}}$.

51. $\dfrac{x - 1 + \frac{6}{x - 6}}{x - 2 + \frac{3}{x - 6}}$.

52. $\dfrac{\frac{x^2 + xy + y^2}{x^3 + y^3}}{\frac{x^3 - y^3}{x^2 - xy + y^2}}$.

53. $\dfrac{\frac{2}{x - 3} + \frac{2x^2 + 2}{x - 2x^3}}{\frac{1}{x - 3} - \frac{1}{1 - 2x^2}}$.

54. $a^2 + 3 + 2a + \dfrac{(a^2 + 3)(a^2 - 6)}{2a - 3 - a^2}$.

32. Exponents. Thus far we have dealt with positive integral exponents only and have obtained the laws relating to their use from our study of multiplication and division. These laws may be illustrated as follows:

$$(x^3)(x^2) = x^{3+2} = x^5.$$
$$\frac{x^3}{x^2} = x^{3-2} = x^1.$$
$$(x^3)^2 = x^{(3)(2)} = x^6.$$
$$(xyz)^2 = x^2y^2z^2.$$
$$\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}.$$
$$\sqrt[2]{x^3} = x^{3/2}.$$

A more general statement for each one would be as follows:

$$(x^m)(x^n) = x^{m+n}.$$
$$\frac{x^m}{x^n} = x^{m-n}.$$
$$(x^m)^n = x^{mn}.$$
$$(xyz)^m = x^my^mz^m.$$
$$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}.$$
$$\sqrt[n]{x^m} = x^{m/n}.$$

Now, it is necessary to extend these laws to include other exponents than positive integral ones. These exponents would take in the zero exponent, the negative exponent, and the fractional exponent. Logarithms, another form of exponents, will be treated later.

33. The zero exponent.

$$\frac{x^m}{x^n} = x^{m-n}.$$

Letting $\qquad n = m,$

then $$\frac{x^m}{x^n} = \frac{x^m}{x^m} = x^{m-m} = x^0.$$

But $$\frac{x^m}{x^m} = 1.$$

Therefore, $$x^0 = \frac{x^m}{x^m} = 1.$$

Thus we may say that any number, not zero, raised to the zero power is equal to 1.

34. Negative exponent.

$$\frac{x^2}{x^4} = x^{2-4} = x^{-2}.$$

But $$\frac{x^2}{x^4} = \frac{1}{x^2}.$$

Therefore, $$x^{-2} = \frac{1}{x^2}.$$

Thus, we may say that a number or factor with a negative exponent is equal to 1 divided by the factor with a positive exponent.

Examples:

$$x^{-3} = \frac{1}{x^3}.$$

$$y^{-6} = \frac{1}{y^6}.$$

35. Fractional exponent.

Since $$\sqrt[n]{x^m} = x^{m/n},$$

then if m and n may take any values, we get what is called a fractional exponent. Thus,

$$\sqrt[3]{m^2} = m^{2/3} \quad \text{and} \quad \sqrt{m} = m^{1/2}.$$

Therefore, a fractional exponent is one that indicates a root. The numerator is the exponent of a power and the denominator is the index of the root.

$$6^{2/3} = \sqrt[3]{6^2} = \sqrt[3]{36},$$
$$x^{5/8} = \sqrt[8]{x^5}.$$

36. Radicals. The indicated root of a number is called a *radical;* the quantity whose root is required is called the *radicand.* Thus, $\sqrt{6}$, $\sqrt[3]{7}$, $\sqrt{a^2 + b^2}$ are radicals whose radicands are 6, 7 and $a^2 + b^2$ respectively. The small figure written in the $\sqrt{\ }$ of the radical sign is called the index of the radical and indicates the required root. If no figure is written in the sign, then the second or square root is understood. It is evident then that a radical is simply another way of expressing a fractional exponent and follows the laws of exponents.

Thus, $$x^{m/n} = \sqrt[n]{x^m},$$
$$x^{1/n}y^{1/n} = (xy)^{1/n} = \sqrt[n]{xy},$$
$$\frac{x^{1/n}}{y^{1/n}} = \left(\frac{x}{y}\right)^{1/n} = \sqrt[n]{\frac{x}{y}}.$$

The following definitions are helpful in the study of radicals:

1. A rational number is one that may be expressed as an integer or as a fraction with integral terms. Examples: 2, $\frac{1}{3}$, 0.33, $\sqrt{49}$.

2. An irrational number is one that cannot be expressed as an integer or as a fraction with integral terms. Examples: $\sqrt{\frac{1}{2}}$, $1+\sqrt{2}$, $\sqrt{1+\sqrt{2}}$.

3. An expression is called a surd when the indicated root of a rational number cannot be exactly obtained. Examples: $\sqrt{2}$, $\sqrt{5}$.

$\sqrt{1+\sqrt{2}}$ is not a surd since $1+\sqrt{2}$ is not rational.

4. The order of a radical or surd is shown by the index of the root or the denominator of the fractional exponent.

A quadratic surd is a surd of the second order.

A cubic surd is a surd of the third order.

37. Simplification of radicals. A radical may be changed as to the form in which it is written without changing its value. There are many reasons why such changes are desirable. For example, the addition or subtraction of fractions with different radicals in the denominators is made easier by changing the fractions so that the same radicals appear in the denominators.

Radicals may be changed in form in any one of the following ways: factoring out the perfect squares; putting all factors under the radical sign; removing a radical from the denominator; or reducing two or more radicals to the same order.

To factor out the perfect squares, the radicand is separated into two factors one of which is the greatest perfect square it contains. The square root of this factor is taken and written as a coefficient of the other factor as a radical. This reduces the original radical to its simplest form.

EXAMPLE 3–48. Reduce $\sqrt{24x^5}$ to its simplest form.

Solution: $\sqrt{24x^5} = \sqrt{(4x^4)(6x)} = 2x^2\sqrt{6x}$.

To put all factors under the radical sign, the coefficient of the radical is first raised to the power indicated by the radical and then multiplied by the term under the radical to form a new radicand.

EXAMPLE 3–49. Change $2a\sqrt{3}$ to a factor under the radical sign.

Solution: $2a\sqrt{3} = \sqrt{(2a)^2(3)} = \sqrt{(4a^2)(3)} = \sqrt{12a^2}$.

Note that this is the reverse of the process in Example 3–48.

To remove a radical from the denominator, the numerator and denominator are multiplied by the denominator, thereby making the denominator a perfect square, whose exact square root can be taken. This process is called *rationalizing the denominator*.

EXAMPLE 3–50. Remove the radical from the denominator in $\sqrt{\dfrac{4x^3}{y}}$.

Solution:

$$\sqrt{\frac{4x^3}{y}} = \sqrt{\left(\frac{4x^3}{y}\right)\left(\frac{y}{y}\right)} = \sqrt{\frac{4x^3y}{y^2}} = \sqrt{\frac{4x^2}{y^2}(xy)} = \frac{2x}{y}\sqrt{xy}.$$

To reduce radicals to the same order, they are written first with fractional exponents having a common denominator. Then each radical is rewritten with the denominator as a root and the numerator as a power of the radicand.

EXAMPLE 3–51. Reduce $\sqrt{2}$; $\sqrt[3]{4}$; and $\sqrt[4]{5}$ to the same order.

Solution:
$$\sqrt{2} = 2^{1/2} = 2^{6/12} = \sqrt[12]{2^6} = \sqrt[12]{64},$$
$$\sqrt[3]{4} = 4^{1/3} = 4^{4/12} = \sqrt[12]{4^4} = \sqrt[12]{256},$$
$$\sqrt[4]{5} = 5^{1/4} = 5^{3/12} = \sqrt[12]{5^3} = \sqrt[12]{125}.$$

EXAMPLE 3–52. Reduce to radicals of the same order: $\sqrt{xy}$; $\sqrt[3]{x^2y}$; $\sqrt{5a}$.

Solution:
$$\sqrt{xy} = (\sqrt{x})(\sqrt{y}) = x^{1/2}y^{1/2} = x^{3/6}y^{3/6} = (\sqrt[6]{x^3})(\sqrt[6]{y^3}) = \sqrt[6]{x^3y^3},$$
$$\sqrt[3]{x^2y} = (\sqrt[3]{x^2})(\sqrt[3]{y}) = x^{2/3}y^{1/3} = x^{4/6}y^{2/6} = (\sqrt[6]{x^4})(\sqrt[6]{y^2}) = \sqrt[6]{x^4y^2},$$
$$\sqrt{5a} = (\sqrt{5})(\sqrt{a}) = 5^{1/2}a^{1/2} = 5^{3/6}a^{3/6} = (\sqrt[6]{5^3})\sqrt[6]{a^3} = \sqrt[6]{125a^3}.$$

38. Addition and subtraction of radicals. *Radicals can be added or subtracted only if they are of the same order and have the same radicand.* It should be clearly understood that both conditions must be fulfilled; otherwise addition or subtraction can be indicated only.

EXAMPLE 3–53. Simplify $\sqrt{18} + \sqrt{98} - \sqrt{128}$.

Solution:
$$\sqrt{18} = \sqrt{(9)(2)} = 3\sqrt{2},$$
$$\sqrt{98} = \sqrt{(49)(2)} = 7\sqrt{2},$$
$$\sqrt{128} = \sqrt{(64)(2)} = 8\sqrt{2}.$$

Therefore,
$$\sqrt{18} + \sqrt{98} - \sqrt{128} = 3\sqrt{2} + 7\sqrt{2} - 8\sqrt{2} = 2\sqrt{2}.$$

EXAMPLE 3–54. Simplify $\sqrt{4abc} - \sqrt[4]{a^2b^2c^2} + \sqrt[8]{256a^4b^4c^4}$.

Solution:
$$\sqrt{4abc} = 4^{1/2}a^{1/2}b^{1/2}c^{1/2} = 2a^{1/2}b^{1/2}c^{1/2},$$
$$\sqrt[4]{a^2b^2c^2} = a^{2/4}b^{2/4}c^{2/4} = a^{1/2}b^{1/2}c^{1/2},$$
$$\sqrt[8]{256a^4b^4c^4} = 256^{1/8}a^{4/8}b^{4/8}c^{4/8} = 2a^{1/2}b^{1/2}c^{1/2}.$$

Therefore,
$$\begin{aligned}\sqrt{4abc} - \sqrt[4]{a^2b^2c^2} + \sqrt[8]{256a^4b^4c^4} &= 2a^{1/2}b^{1/2}c^{1/2} - a^{1/2}b^{1/2}c^{1/2} + 2a^{1/2}b^{1/2}c^{1/2} \\ &= 3a^{1/2}b^{1/2}c^{1/2} \\ &= 3(abc)^{1/2} \\ &= 3\sqrt{abc}.\end{aligned}$$

39. Multiplication of radicals. To multiply radicals they must be reduced to the same order first. Then the coefficients are multiplied to give the coefficient of the product, and the radicands are multiplied to give the radical factor of the product. If the radicals are of the same order, the multiplications of the coefficients and of the radicands can be performed at once. If two radicands are the same and the radicals are of the

same order, then their multiplication results in the given radicand. Thus, $(\sqrt{x})(\sqrt{x}) = x$, which can be checked by the use of fractional exponents.

$$(\sqrt{x})(\sqrt{x}) = (x^{1/2})(x^{1/2}) = x^{1/2+1/2} = x^1 = x.$$

Note: The above does not apply for the square root of a negative number. This condition will be discussed in Article 42.

EXAMPLE 3–55. Multiply $2\sqrt{3}$ by $3\sqrt{5}$.

Solution: Since both are of the same order, i.e., both are square roots, the multiplication can be performed at once without further change.

$$(2\sqrt{3})(3\sqrt{5}) = (2)(3)\sqrt{(3)(5)} = 6\sqrt{15}.$$

EXAMPLE 3–56. Multiply $\sqrt{2}$ by $\sqrt[3]{3}$.

Solution: These must be reduced to the same order before multiplication can be performed.

$$\sqrt{2} = 2^{1/2} = 2^{3/6} = \sqrt[6]{(2)^3} = \sqrt[6]{8},$$
$$\sqrt[3]{3} = 3^{1/3} = 3^{2/6} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$

Then $$(\sqrt{2})(\sqrt[3]{3}) = (\sqrt[6]{8})(\sqrt[6]{9}) = \sqrt[6]{72}.$$

If the radicals to be multiplied are in the polynomial form, the multiplication is performed just the same as for any polynomials.

EXAMPLE 3–57. Multiply $\sqrt{2a} + \sqrt{3b}$ by $\sqrt{3a} - \sqrt{2b}$.

Solution:

$$\begin{array}{l}
\sqrt{2a} + \sqrt{3b} \\
\underline{\sqrt{3a} - \sqrt{2b}} \\
\sqrt{6a^2} + \sqrt{9ab} \\
\underline{\qquad\quad - \sqrt{4ab} - \sqrt{6b^2}} \\
a\sqrt{6} + 3\sqrt{ab} - 2\sqrt{ab} - b\sqrt{6} = a\sqrt{6} + \sqrt{ab} - b\sqrt{6} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad = (a - b)\sqrt{6} + \sqrt{ab}.
\end{array}$$

40. Division of radicals. To divide radicals, they are first reduced to the same order. Then the quotient of the coefficients and the quotient of the radicands are obtained and a new radical written with this new coefficient and new radicand. The new radical may be reduced to its simplest form.

EXAMPLE 3–58. Divide $6\sqrt[3]{y^2}$ by $3\sqrt{x}$.

Solution: $$\frac{6\sqrt[3]{y^2}}{3\sqrt{x}} = \frac{(6)(y^{2/3})}{(3)(x^{1/2})} = \frac{6y^{4/6}}{3x^{3/6}} = \frac{6\sqrt[6]{y^4}}{3\sqrt[6]{x^3}} = 2\sqrt[6]{\frac{y^4}{x^3}}.$$

The denominator may be rationalized in the final form. Thus,

$$2\sqrt[6]{\frac{y^4}{x^3}} = 2\sqrt[6]{\frac{y^4}{x^3}\left(\frac{x^3}{x^3}\right)} = 2\sqrt[6]{\frac{y^4x^3}{x^6}} = \frac{2}{x}\sqrt[6]{y^4x^3}.$$

It is frequently necessary to evaluate a fractional expression containing a radical in the denominator. Since it is rather difficult and cumbersome to work out the value for the radical and then perform the division, a process

for removing the radical from the denominator without changing the value of the expression is used. This process is called *rationalization* and consists fundamentally of multiplying both numerator and denominator by a factor that will remove the radical from the denominator and leave an integer. The simple case was illustrated in Example 3–47, but it is also important to consider expressions in the binomial form.

A binomial containing an integer and a quadratic surd is called a *binomial quadratic surd*, as, for example, $4+\sqrt{6}$. The expressions $a+\sqrt{b}$ and $a-\sqrt{b}$ are binomial quadratic surds differing only in the sign between the terms. Such expressions are called *conjugates;* that is, $a+\sqrt{b}$ is the conjugate of $a-\sqrt{b}$ and $a-\sqrt{b}$ is the conjugate of $a+\sqrt{b}$. The term conjugate is used considerably in certain types of electrical circuits and should be remembered.

To divide a quantity by a binomial quadratic surd, the denominator is rationalized by multiplying both numerator and denominator by the conjugate of the denominator. This is equivalent to multiplying both numerator and denominator by the denominator with the sign between the terms changed. Such a multiplication results in an expression in the denominator that will be the difference of two squares because, as we have learned in multiplication, $(x+y)(x-y) = x^2 - y^2$. Therefore, the new denominator will contain no radical term and the division can be performed easily.

EXAMPLE 3–59. Rationalize $\dfrac{3}{4-\sqrt{6}}$.

Solution:

$$\begin{aligned}\frac{3}{4-\sqrt{6}} &= \frac{3}{4-\sqrt{6}}\,\frac{4+\sqrt{6}}{4+\sqrt{6}},\\ &= \frac{3(4+\sqrt{6})}{(4-\sqrt{6})(4+\sqrt{6})} = \frac{12+3\sqrt{6}}{4^2-(\sqrt{6})^2},\\ &= \frac{12+3\sqrt{6}}{16-6} = \frac{12+3\sqrt{6}}{10} = 1.2+0.3\sqrt{6}.\end{aligned}$$

It should be noted that the sign between the terms of the denominator must be changed to obtain the multiplying factor. Otherwise, a radical will still appear in the new denominator and the work will have been for naught. Thus in Example 3–56 had we multiplied by $4-\sqrt{6}$ there would have resulted

$$\frac{(3)(4-\sqrt{6})}{(4-\sqrt{6})(4-\sqrt{6})} = \frac{12-3\sqrt{6}}{16-8\sqrt{6}+6} = \frac{12-3\sqrt{6}}{22-8\sqrt{6}},$$

which is even more complicated than the original expression. It must be remembered also that only the sign between the terms in the denominator is changed to obtain the multiplying factor. Any signs within a term are not to be changed.

EXAMPLE 3–60. Rationalize $\dfrac{x-\sqrt{x^2+1}}{x+\sqrt{x^2+1}}$.

Solution:

$$\begin{aligned}\frac{x-\sqrt{x^2+1}}{x+\sqrt{x^2+1}} &= \left[\frac{x-\sqrt{x^2+1}}{x+\sqrt{x^2+1}}\right]\left[\frac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\right],\\ &= \frac{x^2-2x\sqrt{x^2+1}+(\sqrt{x^2+1})^2}{x^2-(\sqrt{x^2+1})^2},\\ &= \frac{x^2-2x\sqrt{x^2+1}+x^2+1}{x^2-(x^2+1)},\\ &= \frac{2x^2-2x\sqrt{x^2+1}+1}{x^2-x^2-1},\\ &= 2x\sqrt{x^2+1}-2x^2-1.\end{aligned}$$

Note: See Article 43 for a discussion of the square root of a negative number.

41. Powers and roots in radicals. When simplifying radicals that contain powers or roots, it is generally more convenient first to change to fractional exponents.

EXAMPLE 3–61. Simplify $(2\sqrt[5]{x^4})^2$.

Solution: $(2\sqrt[5]{x^4})^2 = 4(x^{4/5})^2 = 4x^{8/5} = 4\sqrt[5]{x^8}$,
$= 4\sqrt[5]{(x^5)(x^3)} = 4x\sqrt[5]{x^3}$.

EXERCISE 3–6

By means of negative exponents, write each of the following without using the fractional form. Do not change the numerical coefficient.

1. $\dfrac{5ax^3}{x^2y^3}$.
2. $\dfrac{2ab^2}{x^3y^9}$.
3. $\dfrac{4b^2c}{b^4c^6x^2}$.
4. $\dfrac{bx^4}{4ab^2}$.
5. $\dfrac{3a+4b^2}{x-y}$.
6. $\dfrac{7x^3y^2}{x^3y-y^2}$.

Express the following without using negative exponents:

7. $ax^{-3}y^{-2}$.
8. $xy^{-a}t^{-1}$.
9. $x^{-3a}y^{-2b}t^{-1}$.
10. $\dfrac{x^2y^{-3}}{4}$.
11. $z^{-3}a^4b^{-3a}$.
12. $2^{-2}x^{-3}y^{-4}$.
13. $4a^2b^{-3}$.
14. $(a^{-3}+b^{-3})^2$.
15. $(x^{-2}y^{-2})^{-2}$.
16. $\dfrac{(a^{-3}b^{-4})^{-2}}{x^{-2}y^{-3}}$.

Place all factors under the radical sign in the following:

17. $2\sqrt{5}$.
18. $b^2\sqrt{a^3}$.
19. $(a-b)\sqrt{\dfrac{1}{a-b}}$.
20. $a\sqrt[3]{(a+b)}$.

Give the equivalent of the following, using fractional exponents:

21. $\sqrt{a^3bc}$.
22. $\sqrt[5]{a^3y^6x^7}$.
23. $\sqrt[4]{a^2b^4c^6}$.
24. $\sqrt{x^2a^2y}$.
25. $\sqrt[3]{9a^4b^2}$.
26. $\sqrt{27x^4-9x^3y^2}$.

Simplify:

27. $\sqrt[3]{125(x+y)^4}$.

28. $\sqrt{18a-9}$.

29. $\sqrt{27(x+y)(a-b)^3}$.

30. $\sqrt{27}+2\sqrt{48}-3\sqrt{75}$.

31. $\sqrt{a^3-a^2b}-\sqrt{ab^2-b^3}-\sqrt{(a+b)(a^2-b^2)}$.

32. $\sqrt{(a^2b^3c^5)^3}$.

33. $\sqrt{(x+y)^2}$.

34. $(\sqrt[3]{bc^3d^5})^3$.

35. $(3\sqrt{5}-2\sqrt{3})(3\sqrt{5}+2\sqrt{3})$.

36. $(3x+x\sqrt{3})(3x-x\sqrt{3})$.

37. $\dfrac{\sqrt{x-3}-3}{\sqrt{x-3}+3}$.

38. $\dfrac{1}{a\sqrt{a+b}+\sqrt{y}}$.

39. $\dfrac{2\sqrt{x}+6}{3\sqrt{x}-4}$.

40. $\dfrac{4\sqrt{y}-\sqrt{x}}{3\sqrt{y}-2\sqrt{x}}$.

41. $\sqrt[3]{128(a+b)^5}$.

42. $\sqrt{32a^2-16a}$.

43. $\sqrt{8(a^2+b^2)^2(a-b)}$.

44. $\dfrac{\sqrt{b}-2\sqrt{a}}{2\sqrt{a}+3\sqrt{b}}$.

45. $\dfrac{a-\sqrt{a-2}}{a+\sqrt{a-2}}$.

46. $\dfrac{\sqrt{b}-4\sqrt{a}}{3\sqrt{b}-\sqrt{a}}$.

42. Rotation of numbers by multiplication by (-1). In Article 3 numbers were represented on a scale with zero at the center, positive numbers extending to the right and negative numbers extending to the left. It was shown that the addition and subtraction of numbers could be represented along this scale. Thus, to add 3 to 2, we start at $+2$, count three units to the right, and obtain $+5$. To add -3 to -2, we start at -2, count three units to the left, and obtain -5. Again, since subtracting a number is the same as adding the number with its sign changed, we can perform subtraction along this scale. To subtract -4 from $+3$, we change the sign of -4 to $+4$ and add to $+3$. Therefore, we start at $+3$, count 4 units to the right, and obtain $+7$. To subtract $+3$ from -4, we change the sign of $+3$ to -3 and add to -4. Therefore, we start at -4, count three units to the left, and obtain -7. It is worthy of note that the addition of a positive number always requires moving toward the right of the scale and the addition of a negative number always requires moving toward the left of the scale.

The representation of numbers along such a scale can also be used for the multiplication and division of numbers.

Multiplication may be defined as the process of adding the multiplicand as many times as there are units in the multiplier and is therefore only an extension of addition. Thus, 3×4 means that the multiplicand, 4, is taken three times additively, since the multiplier, 3, has three units. Then 3×4 is the same as $4+4+4$ and gives the same result.

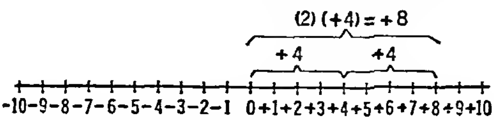


Fig. 3-3.

The multiplication of 2×4 is shown in Fig. 3–3. 4 is taken twice additively because there are two units in the multiplier. Thus, $2 \times 4 = 4 + 4$. The multiplication of a negative number by a positive number is shown in Fig. 3–4. -2 is taken four times additively in the negative direction because there are four units in the multiplier.

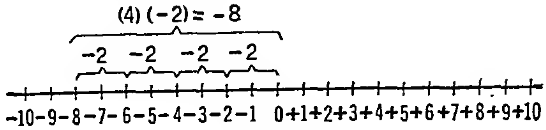


Fig. 3–4.

It is evident then that a positive multiplier changes only the magnitude of the number being multiplied. However, there are negative multipliers to consider, as, for example, the problem of $(-2)(+4)$. Here it is not possible to take the multiplicand, 4, additively -2 times, and so it will be necessary to make some change in the form of the multiplication. Such a change can be accomplished because $(-2) \times 4 = -8$ and $2 \times 4 \times (-1) = -8$. Therefore, we shall first multiply 2×4 to obtain 8 and then multiply by -1 to obtain -8. The multiplication of 2×4 is shown in Fig. 3–3 where the answer, $+8$, is represented as a number 8 units in length and

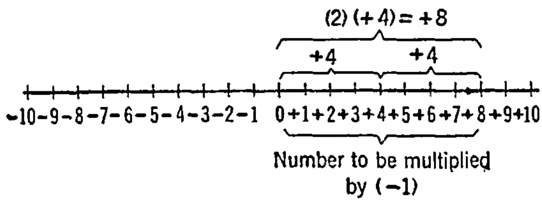


Fig. 3–5.

directed toward the right. Now, when $+8$ is multiplied by -1, the answer, -8, must be represented as a number 8 units in length and directed toward the left. Therefore, it is plain that the multiplication by -1 rotates a number so that it will be directed in the opposite direction from its original position. Illustrations are given by Fig. 3–5 and Fig. 3–6.

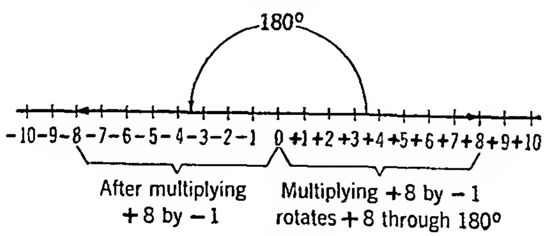


Fig. 3–6.

All the foregoing representations apply to division as well as to multiplication since the laws of signs are the same. The concept of rotating a number to an exactly opposite direction from its original position through its multiplication by -1 is a very important one because it is used

extensively in electrical circuits. The number -1 is considered as simply an operator when used as a multiplier or divisor for the purpose of rotation.

43. Square roots of negative numbers. In the removal of factors from radicands, we have dealt only with positive numbers. Further, we have found the square roots of positive numbers only. The questions then arise as to what to do with negative radicands and what the meaning of the square root of a negative number is.

According to the laws of multiplication, there is no number that when multiplied by itself or raised to an even power it will produce a negative result. For instance, it is impossible to extract the square root of a negative number, because any real number when squared gives a positive result. Thus, the square root of a negative number has come to be known as an imaginary number and it has become necessary to introduce a new type of number to represent these imaginary numbers.

This new type of number derives from the fact that any imaginary number can be expressed as the product of a positive number and $\sqrt{-1}$. For example,

$$\sqrt{-16} = \sqrt{(-1)(16)} = (\sqrt{-1})\sqrt{16} = (\sqrt{-1})(4),$$

also
$$\sqrt{-10} = \sqrt{(-1)(10)} = (\sqrt{-1})(\sqrt{10}) = (\sqrt{-1})(3.16).$$

This $\sqrt{-1}$ is indicated by the letter i in most mathematics texts, but in the engineering field the letter i is used to indicate electrical current. Therefore, to avoid confusion, engineers use the letter j to denote $\sqrt{-1}$ and we shall follow this practice. This letter j is commonly known as the *operator* j, and is used extensively in alternating-current circuits.

EXAMPLE 3-62. Rewrite the following, using the operator j: $\sqrt{-25}$; $\sqrt{-x^2}$; $-\sqrt{-9x^2}$.

Solution:

$$\sqrt{-25} = \sqrt{(-1)(25)} = (\sqrt{-1})(\sqrt{25}) = j5,$$
$$\sqrt{-x^2} = \sqrt{(-1)(x^2)} = (\sqrt{-1})\sqrt{x^2} = jx,$$
$$-\sqrt{-9x^2} = -\sqrt{(-1)(9x^2)} = -(\sqrt{-1})(\sqrt{9x^2}) = -j3x.$$

When multiplying or dividing radicals composed of even roots (i.e., square root, fourth root, etc.) the radicals must be changed into a form containing the operator j before the problem can be worked. Thus, $(\sqrt{-2})(\sqrt{-3})$ would appear to be $\sqrt{6}$ if we followed the laws of exponents for positive numbers, but this answer is incorrect. Actually $(\sqrt{-2})(\sqrt{-3}) = -\sqrt{6}$ and is proved thus:

$$(\sqrt{-2}) = (\sqrt{2})(\sqrt{-1}) = \sqrt{2}j,$$

and
$$(\sqrt{-3}) = (\sqrt{3})(\sqrt{-1}) = \sqrt{3}j.$$

Then
$$(\sqrt{-2})(\sqrt{-3}) = (\sqrt{2}j)(\sqrt{3}j) = \sqrt{6}j^2.$$

But
$$j^2 = -1, \text{ as will be proved later.}$$

Therefore,
$$\sqrt{6}j^2 = (\sqrt{6})(-1) = -\sqrt{6}.$$

44.. Representation of imaginary numbers. In Article 38, it was shown that $\sqrt{x}$ is such a number that $(\sqrt{x})(\sqrt{x}) = x$. Therefore, it follows that the square root of any number when multiplied by itself results in the given number. From this definition, it becomes evident that $(\sqrt{-1})(\sqrt{-1}) = -1$ rather than $+1$ as might seem to be the case. Also in Article 41, we have seen that multiplication or division by -1 rotates a number through 180°. Obviously then, if we multiply a number by -1 two times, it will be rotated through 360° or back to its starting position. This is equivalent to multiplying the number by $+1$.

With these considerations in mind, it is logical to suppose that multiplication of a number by $\sqrt{-1}$ or j should rotate the number halfway to 180° or to 90°, because multiplying two times by j would rotate the number 180° from its original position. This is true from the fact that $(\sqrt{-1})(\sqrt{-1}) = -1$, as we have indicated previously.

Thus, $$(\sqrt{-1})(\sqrt{-1}) = -1,$$
or $$(j)(j) = -1,$$
and $$j^2 = -1.$$

Therefore, it is evident that all directed numbers that are affected by the operator j will lie along a line drawn at 90° to the line used to represent positive and negative numbers. This is illustrated in Fig. 3–7.

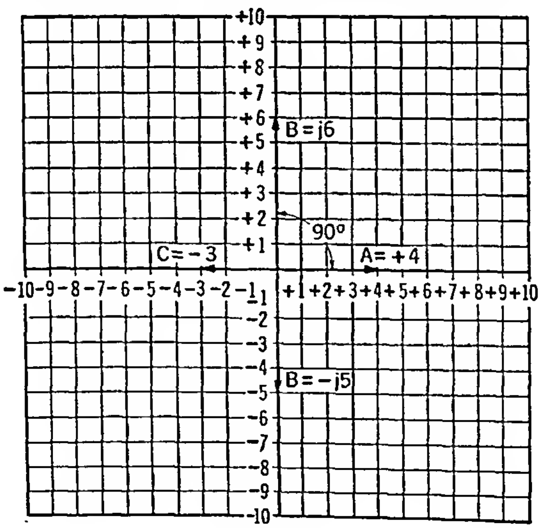


Fig. 3–7.

The horizontal line on which the positive and negative numbers are plotted is called the *axis of reals* and the vertical or 90° line on which the j values are plotted is called the *axis of imaginaries*.

By common practice, numbers that are affected by $+j$ are considered as rotated 90° in a counterclockwise direction from the positive axis of reals, while numbers that are affected by $-j$ are considered as rotated 90°

in a clockwise direction from the positive axis of reals. These are indicated in Fig. 3–8.

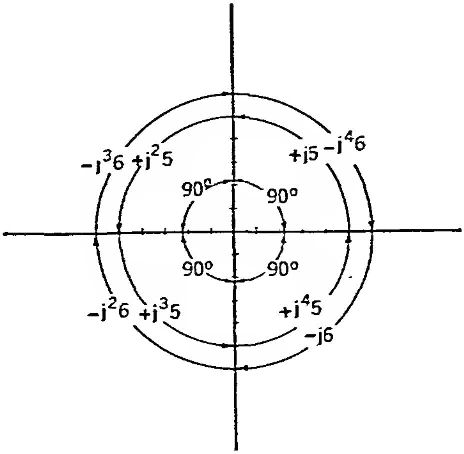


Fig. 3–8.

Successive multiplication by each $+j$ will rotate a number 90° in a counterclockwise direction and by each $-j$ will rotate a number 90° in a clockwise direction. These may be verified thus:

For successive multiplications by $+j$,

$$(\sqrt{-1})(\sqrt{-1}) = -1,$$

or
$$(j)(j) = j^2.$$

Therefore, $j^2 = -1$ (rotation 180° counterclockwise);

also
$$(\sqrt{-1})(\sqrt{-1})(\sqrt{-1}) = (-1)\sqrt{-1} = -j,$$

or
$$(j)(j)(j) = j^3,$$

Therefore, $j^3 = -j$ (rotation 90° clockwise or 270° counterclockwise);

also
$$(\sqrt{-1})(\sqrt{-1})(\sqrt{-1})(\sqrt{-1}) = (-1)(-1) = +1,$$

or
$$(j)(j)(j)(j) = j^4.$$

Therefore, $j^4 = +1$ (rotation 360° counterclockwise).

For successive multiplication by $-j$,

$$(-\sqrt{-1})(-\sqrt{-1}) = +(-1) = -1,$$

or
$$(-j)(-j) = +j^2.$$

Then
$$(-j)^2 = -1 \text{ (rotation 180° clockwise);}$$

also
$$(-\sqrt{-1})(-\sqrt{-1})(-\sqrt{-1}) = (-1)(-\sqrt{-1}) = \sqrt{-1},$$

or
$$(-j)(-j)(-j) = (-1)(-j) = +j.$$

Then $(-j)^3 = +j$ (rotation 90° counterclockwise or 270° clockwise).

$$(-\sqrt{-1})(-\sqrt{-1})(-\sqrt{-1})(-\sqrt{-1}) = (-1)(-1) = +1,$$

or
$$(-j)(-j)(-j)(-j) = (-j)^2(-j)^2 = (-j)^4.$$

Then $(-j)^4 = +1$ (rotation 360° clockwise).

These multiplications are all illustrated in Fig. 3–8.

To summarize all this, it is evident that j is simply an operator that is used to rotate a directed number through 90° regardless of the value of that number, with $+j$ denoting counterclockwise rotation and $-j$ denoting clockwise direction.

45. Complex numbers. When a real number and an imaginary number are united by a plus or a minus sign to form a binomial, the resulting expression is called a *complex number*. Thus, $a + jb$, $3 - j4$, and $R + jx$ are complex numbers. The real number and the imaginary number cannot be added or subtracted arithmetically since they represent numbers that are at 90° or right angles to each other, but the addition, subtraction, multiplication, or division of complex numbers can be handled algebraically by treating them as ordinary binomials.

46. Addition and subtraction of complex numbers. To add complex numbers, the real numbers are added together, the j numbers are added together, and the sums obtained are combined to form a new complex number.

To subtract complex numbers, the signs of the subtrahend are changed, then the real numbers are added, the j numbers are added, and the results are combined to form a new complex number.

EXAMPLE 3–63. Add $4 + j6$ and $3 - j2$.

Solution:

$$\begin{array}{r} +4 + j6 \\ +3 - j2 \\ \hline 7 + j4 \end{array}$$

EXAMPLE 3–64. Subtract $2 - j5$ from $6 + j8$.

Solution:

$$\begin{array}{rl} +6 + j8 & \\ +2 - j5 & \\ - \quad + \quad & \leftarrow \text{changed signs} \\ \hline 4 + j13 & \end{array}$$

47. Multiplication of complex numbers. To multiply complex numbers they are treated the same as any binomials. However, it must be remembered that $j^2 = -1$.

EXAMPLE 3–65. Multiply $3 + j7$ by $4 - j5$.

Solution:

$$\begin{array}{l} 3 + j7 \\ 4 - j5 \\ \hline 12 + j28 \\ \qquad - j15 - j^2 35 \\ \hline 12 + j13 - j^2 35 \end{array}$$

But since $j^2 = -1$, the final result will be $12 + j13 - (-1)(35) = 12 + j13 + 35 = 47 + j13$.

48. Division of complex numbers. To divide complex numbers, the complete expression is rationalized to obtain a real number as the divisor and the actual division then is performed if possible. To rationalize, the

numerator and denominator are multiplied by the conjugate of the denominator.

EXAMPLE 3–66. Perform the division in $\frac{5 + j3}{3 - j4}$.

Solution:

$$\frac{5 + j3}{3 - j4} = \left[\frac{5 + j3}{3 - j4}\right]\left[\frac{3 + j4}{3 + j4}\right] = \frac{15 + j29 + j^2 12}{9 - j^2 16}$$
$$= \frac{15 + j29 + (-1)(12)}{9 - (-1)(16)}$$
$$= \frac{3 + j29}{9 + 16} = \frac{3 + j29}{25}$$
$$= 0.12 + j1.16.$$

49. Multiplication and division of complex numbers. The combined operations of multiplication and division of complex numbers are encountered in the solution of alternating current single phase parallel circuits. For instance, the solution for a two-branch parallel circuit can be obtained by dividing the product of two complex numbers by their sum.

EXAMPLE 3–67. Evaluate $\frac{(4 + j5)(5 - j2)}{(4 + j5) + (5 - j2)}$.

Solution:

$$\frac{(4 + j5)(5 - j2)}{(4 + j5) + (5 - j2)} = \frac{20 + j25 - j8 - j^2 10}{9 + j3}$$
$$= \frac{20 + j17 - (-1)(10)}{9 + j3}$$
$$= \frac{30 + j17}{9 + j3}.$$

This form can now be rationalized:

$$\left[\frac{30 + j17}{9 + j3}\right]\left[\frac{9 - j3}{9 - j3}\right] = \frac{270 + j153 - j90 - j^2 51}{81 - j^2 9}$$
$$= \frac{321 + j63}{81 + 9}$$
$$= \frac{321 + j63}{90}$$
$$= 3.57 + j0.7.$$

EXERCISE 3–7

Express the following by use of the operator j:

1. $\sqrt{-16}$.
2. $-\sqrt{-(36x^2)}$.
3. $-2\sqrt{-9}$.
4. $\sqrt{\frac{-25}{4}}$.
5. $-\sqrt{\frac{64}{-81}}$.
6. $\frac{\sqrt{-121R^2}}{R}$.
7. $-7\sqrt{\frac{-625}{49}}$.
8. $\sqrt{\frac{-27}{16}}$.
9. $\sqrt{\frac{25}{-32}}$.
10. $-2\sqrt{\frac{-9}{36}}$.

Find the sum in each of the following:

11. $\begin{array}{r} 8 + j3 \\ 2 - j5 \\ \hline \end{array}$

12. $\begin{array}{r} 36 - j21 \\ -15 - j10 \\ \hline \end{array}$

13. $\begin{array}{r} -10 + j12 \\ -20 - j35 \\ \hline \end{array}$

14. $\begin{array}{r} 11 + j7 \\ -17 + j18 \\ \hline \end{array}$

15. $\begin{array}{r} 15 + j20 \\ 20 + j15 \\ \hline \end{array}$

16. $\begin{array}{r} -21 - j25 \\ 32 + j19 \\ \hline \end{array}$

17 to 22 inclusive. Subtract the lower complex number from the upper in each of Problems 11 to 16 inclusive.

Perform the indicated operations:

23. $(5 + j7)(4 - j3)$.
24. $(6 + j3)(3 + j4)$.
25. $(7 - j5)(9 - j8)$.
26. $(a + jb)(a + jb)$.
27. $(R + jx)(R - jx)$.
28. $(1 + j)(1 - j)$.
29. $\dfrac{1}{1 + j3}$.
30. $\dfrac{5 + j3}{2 - j4}$.
31. $\dfrac{x - jy}{x + jy}$.
32. $\dfrac{1 - j1}{1 + j1}$.
33. $\dfrac{6 - j5}{3 + j7}$.
34. $\dfrac{9 + j7}{2 + j3}$.
35. $\dfrac{2}{2 - jy}$.
36. $\dfrac{8 - j}{1 + j7}$.
37. $\dfrac{(3 - j2)(7 + j4)}{(3 - j2) + (7 + j4)}$.
38. $\dfrac{(12 + j5)(10 - j9)}{(12 + j5) + (10 - j9)}$.
39. $\dfrac{(6 + j5)(8 + j3)}{(6 + j5) + (8 + j3)}$.
40. $\dfrac{(9 - j2)(2 - j11)}{(9 - j2) + (2 - j11)}$.

REVIEW EXERCISE 3-8

Perform the indicated operations in each of the following problems and simplify:

1. $(3x - 2y) - (2x - 3y) - x$.
2. $(9x - 16y) - 2[3x - (5y + 4x) + (x - 3y) + 2]$.
3. $2(2R - E) - 3[2R - 4(R - 3E) - 7E]$.
4. $-[n - 4\{1 - 2(3n - 1) - (2n + 5)\} - 10]$.
5. $2x - [5 - \{2 - 4(x - 1)\} - 2(x + 3) - x]$.
6. $2a - 3[4b + 2(3a + 5b) - 3\{b - 2(4 - a) + 5\} - 5a]$.
7. $3(x - 2y) - 2[2x - 5y - 4\{3x - (x - 2y) + 3y\}]$.
8. $3[2b - 3\{(4a - 3b) - 2(5a - [2a - b] + 3b)\} - a]$.
9. Enclose the last three terms of the following in negative parentheses and the second and third terms in negative parentheses:

$$7x + 6y - 4z - (2x + 3y) + (3x - 2y - 6z).$$

10. Enclose the following in total negative parentheses and the last two terms in negative parentheses:

$$x - (4x + 3y) + (2x - y) - 6x - 7y.$$

Perform the following multiplications:

11. $(4a)(2b)(-3c^2)$.
12. $-4x(x - 3a^2 + 2 - 4ax)$.

13. $5b^2y(2b - 3y + by - 4x^2)$.
14. $6xyz^3(3z + 2x^3y - 5yz^2)$.
15. $-8a^2b^2(4b^3 - 3a^2b^2 - a^3)$.
16. $(4 - 3x)(2x^2 - x + 5)$.
17. $(2x + y)(x^2 - 2xy + 4y^2)$.
18. $(a^2 + 3a - 2)(2a^2 - 5a + 3)$.
19. $(3y^2 - 5y + 2)(5 - 3y)$.
20. $(2r^2 - 3rx + 2x^2)(3x^2 - r^2 + 2rx)$.
21. $(0.2x^4 - 0.3x^3y + 1.5xy^2 - 0.8y^3)(-0.6x^2 + 0.4xy + 0.5y^2)$.
22. $\left(15a^4 - 16a^3b + \frac{2}{3}a^2b^2 - ab^3 + \frac{4}{5}b^4\right)\left(\frac{3}{5}a^3 - 0.8a^2b + \frac{3}{8}b^2\right)$.

23. The parallel sides of a trapezoid are x inches and $(x + 6)$ inches respectively. The altitude is b feet. What is the area in square inches? in square feet?

24. The base of a triangle is y inches and its altitude is $(y + 5)$ inches. What is the area in square inches? in square feet?

25. The area of a triangle is equal to $\sqrt{s(s - a)(s - b)(s - c)}$ where a, b, and c are the three sides and s is one-half the perimeter. If the three sides of a triangle are x inches, $(x - 2)$ inches, and $(x + 3)$ inches respectively, what is the area?

26. The total area of a cylinder is equal to $2\pi r^2 + 2\pi rh$ where r is the radius of the base and h is the altitude. What is the area of a cylinder with a base diameter of x inches and a height of $2x$ inches?

27. The curved surface of a right circular cone, with altitude of h and base radius of r is found by $\pi r\sqrt{r^2 + h^2}$. Determine the surface of a cone whose base diameter is x inches and whose height is $\sqrt{6}\,x$ inches.

28. The perimeter of an ellipse whose semiaxes are a and b is given by:

$$2\pi\sqrt{\frac{a^2 + b^2}{2}}.$$

If $a = x$ and $b = \frac{x}{2}$ what is the perimeter?

29. The volume of the frustrum of a right cone, with radius of base R, radius of top r, and altitude h, is:

$$\frac{\pi h}{3}(r^2 + rR + R^2).$$

Find the volume of the frustrum of a cone if $r = x$, $R = 2x$ and $h = 3x$.

30. The curved surface of the frustrum of the right cone is $\pi(r + R)\sqrt{h^2 + (R - r)^2}$ where r, R and h have the same meaning as in Problem 29. Determine the curved surface of the frustrum of the right cone of Problem 29.

Expand the following by the binomial theorem:

31. $(a + bc)^5$
32. $(xy - ab)^4$
33. $[(a + b) - c]^5$
34. $[x - (y + b)]^4$
35. $[(a + b) - (m - n)]^4$
36. $[(a + b + c) + (x + y + z)]^3$
37. $(x^2 - y^2)^4$
38. $(2x + 3y)^3$
39. $(a + 2b - 3)^4$
40. $[(3a + 5b)^2 - c]^3$

Perform the indicated divisions in the following, using synthetic division where desirable:

41. $\dfrac{15x^3 + 9x^2 - 6x + 12}{3}$.
42. $\dfrac{6m^2n^2 - 2mn + 4m}{2m}$.
43. $\dfrac{18x^4 + 36x^2y - 27y^3 + 45y^2}{-9xy}$.
44. $\dfrac{21x^3y^2 + 28xy^3 - 42x^2y^2}{7xy^2}$.
45. $\dfrac{3a^3 + 5a^2 - 2a - 6}{a - 1}$.
46. $\dfrac{5x^5 + 8x^4 + 2x^3 + x^2 + x - 1}{x + 1}$.
47. $\dfrac{3x^4 + 4x^3 - 3x^2 - 4}{x + 2}$.
48. $\dfrac{x^4 - 3x^3 - 9x^2 + 16x - 6}{x^2 + 2x - 2}$.
49. $\dfrac{\frac{x^3}{3} - \frac{x^2}{3} - \frac{7x}{9} + \frac{2}{9}}{\frac{x}{3} - \frac{2}{3}}$.

Hint: Multiply both numerator and denominator by 3; then use synthetic division.

50. $\dfrac{2x^3 - 9x^2y + 16xy^2 + 6y^3}{2x - 3y}$.
51. $\dfrac{6x^5 + 9x^4 - 3x + 4}{3x^2 + x - 2}$.
52. $\dfrac{6x^4 - 20x^3 + 2x^2 - 3x + 8}{3x + 2}$.

Factor each of the following into its prime factors:

53. $6x - 18xy$.
54. $5x^2y + 15xy^2 - 20xy$.
55. $9abc - 18a^2bc + 27ab^2c - 36abc^2$.
56. $mr - nr - ms + ns$.
57. $-3x + 15y + 6(x - 5y)^3$.
58. $5x + 3y - 10mx - 6my$.
59. $8ab - 12a^2b$.
60. $(x + y)^3 - x - y$.
61. $(3 - 5a)^2 - 3 + 5a$.
62. $-7x - 21y - 14(x + 3y)^2$.
63. $b + x - (b + x)^2$.
64. $x^2 + xy + 3y + 3x$.

65. $x^3 + x^2 - 4x - 4$.
66. $a^2x + a^2y - 2abx - 2aby + b^2x + b^2y$.

67. $16x^2 - 25y^2$.
68. $27x - 48x^3$.
69. $9 - (a - b)^2$.
70. $(2x - 3y)^2 - 25y^2$.
71. $\dfrac{1}{x^2} - \dfrac{1}{y^2}$.
72. $\dfrac{E^2}{R^2} - \dfrac{64}{I^2}$.
73. $12a^3 - 12a^2 + 3a$.
74. $16a^2 - b^2 + 2bc - c^2$.
75. $x^4 + 7x^2y^2 + 16y^4$.
76. $9x^4 + 12x^2y^2 - 21y^4$.
77. $x^2y^2 - 2xy - 8$.
78. $30R^2 + 23Rx + 3x^2$.
79. $V^2 - 7VR - 60R^2$.
80. $108 - 96j - 35j^2$.

81. $12j^2 - 70j - 150.$
82. $E^2 - 110R^2 - 17RE.$
83. $180(ab)^2 + 9abc - 221c^2.$
84. $m^2 - 10mn + 56n^2.$
85. $9a^2 + 60ab + 99b^2.$
86. $5x^4 + 500 - 145x^2.$
87. $(18.5)^2 - (16.5)^2.$
88. $(20.2)^2 - (12.8)^2.$
89. $8m^4 - 200m^2 + 1152.$
90. $78x^2 - 4x^3 - 378x.$
91. $24x^3 + 210x^2 - 190x.$
92. $a^{2x} - 14 + \dfrac{49}{a^{2x}}.$
93. $R^2 + \dfrac{5R}{12} - \dfrac{1}{6}.$
94. $(a + b)^3 - 27.$
95. $64 + (x + y)^3.$
96. $x^3 + 9x^2 + 20x + 24.$
97. $x^3 - 2x^2 - 23x + 60.$
98. $x^3 + 4x^2 - 4x + 5.$
99. $x^3 - 7x^2 + 16x - 12.$
100. $12x^3 - 20x^2 - 37x + 30.$

(Let $x = \dfrac{2}{3}$ for first factor.)

101. $x^4 - 12x^2 - 13x - 12.$
102. $x^4 + 6x^3 - 8x^2 - 24x + 16.$
103. $18x^3 + 45x^2 - 113x - 20.$
104. $40x^3 - 92x^2 - 232x + 224.$
105. $168x^3 + 102x^2 - 966x + 720.$
106. $240x^3 - 850x^2 - 515x + 300.$

Simplify each of the following:

107. $\dfrac{\dfrac{7x - 21}{x^2 - 6x + 9}}{\dfrac{7x + 21}{x^2 - 9}}.$

108. $\dfrac{\dfrac{(x - y)^2}{x^2 + xy}}{\dfrac{x^2 - y^2}{2x^3y + 4x^2y^2 + 2xy^3}}.$

109. $\dfrac{\dfrac{1}{a^2 + y^2} - \dfrac{1}{a^2 - y^2}}{\dfrac{y}{a + y} - \dfrac{y}{a - y}}.$

110. $\dfrac{\dfrac{4x^4}{y^4} - 4}{\left(1 + \dfrac{x^2}{y^2}\right)\left(1 - \dfrac{y^2}{x^2}\right)}.$

111. $\dfrac{\dfrac{5x - 15}{5x + 10}}{\dfrac{x^2 - 5x + 6}{x^2 - x - 6}}.$

112. $\dfrac{\dfrac{1 - 4x^2}{2x^2 - 5x - 3}}{\dfrac{2 + 2x - 12x^2}{2x^2 - 6x}}.$

113. $\dfrac{\left(\dfrac{y^2 - 10y + 24}{y^2 - y - 12}\right)\left(\dfrac{3y - 27}{2y^2 - 11y - 6}\right)}{\dfrac{3y - 9}{4y^2 - 1}}.$

114. $\dfrac{\dfrac{y - 1}{y + 1} - 1}{\dfrac{2}{y - 1} - \dfrac{2}{y + 1}}.$

115. $\dfrac{\left(\dfrac{a^2 - 4}{a^2 - 4a + 4}\right)\left(\dfrac{a^2 - 3a - 28}{3a^2 - 6a - 105}\right)}{\dfrac{a^2 - 5a - 14}{3a^2 + 9a - 30}}.$

116. $\dfrac{1 - x\left(\dfrac{y - x}{1 + xy}\right)}{x^2 - x\left(\dfrac{x - y}{1 + xy}\right)}$.

117. $\dfrac{1 + x + x^2}{1 + \dfrac{x}{1 - \dfrac{x}{1 + x}}}$.

118. $\dfrac{\dfrac{x + y}{x - y} + \dfrac{x - y}{x + y}}{\dfrac{x - y}{x + y} - \dfrac{x + y}{x - y}}$.

119. $\dfrac{1}{a + \dfrac{1}{a - \dfrac{1}{a}}} + \dfrac{1}{a - \dfrac{1}{a + \dfrac{1}{a}}}$.

120. $\dfrac{\dfrac{1}{R + x} - \left(\dfrac{1}{R - x} - \dfrac{2R}{R^2 - x^2}\right)}{\left(\dfrac{1 + x}{1 - R} - 1\right)\left(\dfrac{1 - R^2}{(R + x)^2}\right)}$.

121. $\dfrac{\dfrac{1 - xy(ax - y)}{a^2}}{\dfrac{xy^2 + 1}{a} - x^2y}$.

122. $\dfrac{\dfrac{2}{1 - 4y^2} - \left(\dfrac{1}{1 + 2y} - \dfrac{1}{1 - 2y}\right)}{\left(\dfrac{1 - 2y}{1 + y} + 4y\right)\left(\dfrac{1 - y^2}{1 - 8y^3}\right)}$.

123. $\dfrac{\dfrac{2a}{a^2 - b^2} - \left(\dfrac{1}{a - b} - \dfrac{1}{a + b}\right)}{\left(1 - \dfrac{1 - b}{a - 1}\right)\left(\dfrac{a^2 - 1}{(a + b)^2}\right)}$.

124. $\dfrac{\dfrac{a^2 + b}{1 - b^2} - \left(\dfrac{1}{1 - b} - 1\right)}{\left(\dfrac{a}{a - b} - 1\right)\left(\dfrac{(a - b)^2}{1 + b}\right)}$.

125. $\dfrac{\left(\dfrac{\dfrac{1}{y} + 1}{\dfrac{1}{y}}\right)\left(\dfrac{\dfrac{1}{y} - y^2}{\dfrac{1}{y}}\right)}{\dfrac{y^2 + \dfrac{1}{y} - 1}{\dfrac{1}{y}}}$.

126. $\left(a - 1 - \dfrac{1}{a - 1 + \dfrac{a}{a - 1}}\right)\left(\dfrac{a^3 + 1}{(a^2 - 1)^2}\right)$.

163. $\dfrac{2\sqrt{a} - 3\sqrt{b}}{3\sqrt{a} + 4\sqrt{b}}$.

164. $\dfrac{\sqrt{x^2+1} - \dfrac{x^2}{\sqrt{x^2+1}}}{\dfrac{1}{\sqrt{x^2-1}}}$.

165. $\dfrac{\dfrac{\sqrt{(x+1)^2 - x}}{\sqrt{x^3-1}}}{\dfrac{\sqrt{x-1}}{\sqrt{x+1}} - \dfrac{\sqrt{x+1}}{\sqrt{x-1}}}$.

166. $\dfrac{(a+1)^{1/2} - (a+1)^{-1/2}}{(a-1)^{1/2} - (a-1)^{-1/2}}$.

167. $\dfrac{\sqrt{\dfrac{4a^4}{(1-a^2)^2} + \dfrac{4a^2}{1-a^2}}}{\dfrac{4a^3 + 4a(1-a^2)}{(1-a^2)^2}}$.

168. $\dfrac{\dfrac{\dfrac{1}{y(1+x)}}{\sqrt{\dfrac{x^2-y^2}{y^2}}} - \dfrac{\dfrac{x}{\sqrt{x^2-y^2}} + 1}{x + \sqrt{x^2-y^2}}}{}$.

169. $\dfrac{\dfrac{x\sqrt{x-1}}{\sqrt{x+1}} - \dfrac{x\sqrt{x+1}}{\sqrt{x-1}}}{\dfrac{-2}{x+1}}$.

170. $\dfrac{(x+y)^{-1/2} + (x-y)^{-1/2}}{(x+y)^{-1/2} - (x-y)^{-1/2}}$.

Perform the indicated operations and write the answer in its simplest form:

171. $(8 + j3)(4 - j7)$.

172. $(3a + j4b)(5a - j6b)$.

173. $(6 - j7)(7 + j5)(3 - j8)$.

174. $\dfrac{5 + j2}{6 - j3}$.

175. $\dfrac{7 - j2}{10 + j4}$.

176. $\dfrac{8 + j2}{1 + j6}$.

177. $\dfrac{9 - j2}{5 - j3}$.

178. Express $\dfrac{3 - \sqrt{-2}}{4 + \sqrt{-2}}$ in the form $a + jb$.

179. Express $\dfrac{2 - j}{2 + 3j}$ in the form $a + jb$.

180. Express $\dfrac{a - \sqrt{-b}}{a + \sqrt{-c}}$ in the form $a + jb$.

181. $\dfrac{(4.6 + j5.3)(7.2 + j8.3)}{(4.6 + j5.3) + (7.2 + j8.3)}$.

182. $\dfrac{(12.2 + j8.7)(6.8 - j15.3)}{(12.2 + j8.7) + (6.8 - j15.3)}$.

183. $\dfrac{(10.5 - j7.2)(9.4 - j6.5)}{(10.5 - j7.2) + (9.4 - j6.5)}$.

184. $\dfrac{(11.4 - j19.1)(17.7 + j32.5)}{(11.4 - j19.1) + (17.7 + j32.5)}$.

Chapter 4

EQUATIONS AND FORMULAS

IN CHAPTER 3 it was shown that letters as well as numbers can be used to represent ideas, and considerable time was given over to a study of the use of both letters and numbers in the fundamentals of algebra. These fundamentals are put into effective use by means of the equation, a very valuable mathematical tool that is used in the solution of all sorts of problems.

1. Simple equations. *An equation is a mathematical statement of equality between two or more quantities.* The sign of equality (=) is used to separate the quantities and to indicate that they are equal. The terms placed at the left of the equality sign are known as the left member of the equation, whereas the terms placed at the right of the equality sign are known as the right member. Thus the following are typical equations because the left member and the right member are equal:

$$6x = 36.$$
$$A = \pi r^2.$$
$$HP = \frac{PLAN}{33{,}000}.$$
$$I = \frac{E}{R}.$$

A numerical equation is one in which all the known quantities are expressed by numbers. Illustration: $5 + 4 = 9$.

A literal equation is one in which some of the known quantities are expressed by letters. Illustration: $A = 3.14r^2$, where r is known.

A fractional equation is one that contains an unknown in some denominator. Illustration: $x + 3 = \frac{10}{x}$.

An integral equation is one that does not contain an unknown number in any denominator. Illustrations: $x + 3 = 8$ and $\frac{2x}{5} + 4 = 6$.

An identical equation is one whose members are identical or of such a form that they can be made identical. Illustrations:

$$6 + 4 = 10; \ (a + b)(a - b) = a^2 - b^2.$$

A conditional equation is one that is true only for certain values of its letters. Illustration: $x + 4 = 9$ is a conditional equation because it is true only when $x = 5$.

A simple equation, sometimes called a linear equation, is one that involves only the first power of any unknown quantities. Illustrations: $2x + 5 = 13$; $x - y + z = 12$.

The solving of an equation involves finding all the values for the unknown that will make the left and right members identical. These values that satisfy the equation are called the *roots of the equation*. For example, in the equation $x + 3 = 7$, 4 is a root because the substitution of 4 for x makes both members of the equation identical. An equation always produces the question, "What value of the unknown will make both members of the equation identical?"

2. Axioms. In the solution of equations, there are several axioms or self-evident truths that will be found helpful. These are enumerated as follows:

1. When equal quantities are added to equal quantities, the sums are equal.

2. When equal quantities are subtracted from equal quantities, the remainders are equal.

3. When equal quantities are multiplied by equal quantities, the products are equal.

4. When equal quantities are divided by equal quantities, the quotients are equal, provided the divisions are not made by zero.

5. Quantities that are equal to the same quantity or to equal quantities are equal to each other.

6. Like powers of equal quantities are equal.

7. Like roots of equal quantities are equal.

8. The whole is equal to the sum of all its parts.

An examination of these axioms will show that any or all of the fundamental operations of addition, subtraction, multiplication, and division can be performed on one side of the equation if the same operation is also performed on the other side. In general, it may be said that any operation may be performed on an equation if the same operation is performed on both members. Therefore, the following rules become apparent:

Rule 1. The same quantity may be added to each member of an equation.

EXAMPLE 4–1. Add 5 to each member of the equation $x + y = 8$.

Solution: $$x + y = 8.$$

Adding 5 to each member, $$x + y + 5 = 8 + 5.$$

Solving, $$x + y + 5 = 13.$$

Rule 2. The same quantity may be subtracted from each member of an equation.

EXAMPLE 4–2. Subtract a from each member of the equation

$$x + a + y = 2b.$$

Solution:

$$x + a + y = 2b,$$
$$x + a + y - a = 2b - a,$$
$$x + y = 2b - a.$$

Rule 3. Each member of an equation may be multiplied by the same quantity.

EXAMPLE 4-3. Multiply each member of the equation

$$\frac{3b - x}{5} = \frac{7a}{3} \text{ by } 2.$$

Solution:
$$\frac{3b - x}{5} = \frac{7a}{3}.$$

Multiplying by 2,
$$\frac{2(3b - x)}{5} = \frac{(2)(7a)}{3}.$$

Simplifying,
$$\frac{6b - 2x}{5} = \frac{14a}{3}.$$

Rule 4. Each member of an equation may be divided by the same quantity.

EXAMPLE 4-4. Divide each member of the equation $3x = y - 6$ by 3.

Solution:
$$3x = y - 6.$$

Dividing by 3,
$$\frac{3x}{3} = \frac{y - 6}{3}.$$

Simplifying,
$$x = \frac{y}{3} - 2.$$

3. Solution of simple equations. In the solution of an equation, a process known as *transposition* is used. It consists in changing terms from one side of the equation to the other side and gives rise to the following procedure:

If the sign of a term is plus on one side of the equation, its sign must be changed to minus when it is transposed to the other side of the equation; likewise, if the sign of a term is minus on one side of the equation, its sign must be changed to plus when it is transposed to the other side. A study of this procedure will show that it is simply another way of stating Rules 1 and 2.

Now, the solution of an equation involves three very important steps, as follows:

1. Collect all terms that contain the unknown in the left member of the equation and all other terms in the right member. When transposing a term from one member to the other, be sure to change its sign.

2. Combine the terms in each member.

3. Divide each member of the equation by the coefficient of the unknown.

EXAMPLE 4-5. Solve the equation $25x - 12 = 20x + 18$.

Solution:

Original equation, $\quad 25x - 12 = 20x + 18.$

Transposing, $\quad 25x - 20x = 18 + 12.$

Combining terms, $\quad 5x = 30.$

Dividing by coefficient of x, $\quad \dfrac{5x}{5} = \dfrac{30}{5};$

or $\quad x = 6.$

It may be necessary in the solution of an equation to change from one side of the equation to the other side terms that are multipliers or dividers. In order to do this, the following procedure is necessary:

If one member of an equation is multiplied by a factor, that factor may be changed to the other side of the equation by dividing the other member by the factor; likewise, if one member of an equation is divided by a factor, that factor may be changed to the other side by multiplying the other member by the factor. This is simply a restatement of Rules 3 and 4.

Example 4–6. Solve the equation $2(x - 1) = 8$.

Solution: Since the factor 2 is a multiplier of the left member, it must become a divisor when changed to the right member. Thus,

$$2(x - 1) = 8$$

becomes

$$x - 1 = \frac{8}{2}.$$

Therefore,

$$x - 1 = 4 \quad \text{or} \quad x = 4 + 1 = 5.$$

It is evident that this procedure is the same as in Rule 4, since each member has been, in effect, divided by 2.

Example 4–7. Solve the equation $\dfrac{x - 1}{3} = 5$.

Solution:

$$\frac{x - 1}{3} = 5,$$

$$x - 1 = (3)(5) = 15,$$

$$x = 15 + 1 = 16.$$

This procedure is the same as Rule 3, since each member has been multiplied by 3.

Often the equation to be solved is in fractional form and a process called "clearing the equation of fractions" must be used. This consists of first changing to a common denominator. Then, with an equation in which the denominator of each member is the same, we can multiply each member by this value to eliminate the denominator or we may drop the denominator entirely, and leave an equation consisting of numerators

Example 4–8. Solve the equation

$$\frac{2y}{5} + \frac{y}{8} - \frac{y}{4} = \frac{11}{40}.$$

Solution: The least common denominator is 40. Then, changing each term to this common denominator, we get

$$\frac{16y}{40} + \frac{5y}{40} - \frac{10y}{40} = \frac{11}{40},$$

or

$$\frac{16y + 5y - 10y}{40} = \frac{11}{40}.$$

Then, since the denominators are the same, we have

$$16y + 5y - 10y = 11.$$

Or multiplying each member by 40 we obtain the same result.

$$\cancel{40}\left(\frac{16y + 5y - 10y}{\cancel{40}}\right) = (\cancel{40})\left(\frac{11}{\cancel{40}}\right),$$

Then $\quad 11y = 11, \quad$ or $\quad y = 1.$

Cross multiplication is a term often used in clearing an equality from the fractional form. The product of the numerator of the first fraction and the denominator of the second fraction of the equality is equal to the product of the denominator of the first fraction and the numerator of the second fraction. Thus, in the equality $\frac{2x - 3}{4} = \frac{x + 2}{3}$, we have $(2x - 3)(3) = (4)(x + 2)$. This is simply giving another name to the statement that the product of the extremes is equal to the product of the means in any proportion since any equality in fractional form is essentially a proportion. Thus, the equality $\frac{2x - 3}{4} = \frac{x + 2}{3}$ might be written $2x - 3:4 = x + 2:3$.

In the solution of an equation the same term may appear in each member of the equation. If this term has the same sign in each member, it may be canceled out and a new equation may be formed omitting it.

EXAMPLE 4–9. Solve for x in the equation $x + a = a + b$.

Solution: Since the term $+a$ appears in each member, it may be canceled out, leaving

$$x + \cancel{a} = \cancel{a} + b,$$

or

$$x = b.$$

Again, in the solution of an equation, the signs of all the terms may be changed without changing the value of the equation or destroying the equality.

EXAMPLE 4–10. Solve for x in the equation $5 - x = 2$.

Solution: Since x is negative, it is better to change all the signs, thereby making x positive.

$$5 - x = 2,$$

or

$$-5 + x = -2.$$

Then

$$x = +5 - 2 = 3.$$

EXERCISE 4–1

1. What must be done with the equation $x + 1 = 7$ to make the left member $x + 4$? State the new equation.
2. What must be done with the equation $x + p = r + n$ to make the right member equal to r? State the new equation.
3. What must be done with the equation $x - r = K$ to make the left member $1 - \frac{r}{x}$? State the new equation.

4. What must be done with the equation $\frac{x}{5} + y = 10$ to make the right member 50? State the new equation.

5. If $x - \frac{1}{2} = 3.5$, find $x - 1$. 6. If $x + \frac{1}{2} = 9$, find $x + 1$.

Solve the following equations for the unknown:

7. $10x - 8 = 2x + 16$.
8. $14x - 5x - 15 = 25 + 9 + 2x$.
9. $1.2y + (1.0y + 0.5y) - 14.5 = 0.3y + 0.4y - 2.5 - 7.5$.
10. $2R + 12 + 5R - 6 = 7R + 3 + 4R - (2R + 5)$.

11. $\frac{16Z - 3}{7} = \frac{3Z + 7}{2}$.
12. $\frac{8y - 1}{9} = \frac{9y}{8} - \frac{y}{4}$.
13. $2\frac{2}{3} - \frac{R}{11} + \frac{2R}{3} = 1 + \frac{8R}{11}$.
14. $2x + \frac{x}{7} = 30$.
15. $\frac{4y}{9} + \frac{6y}{27} - \frac{4y}{3} = 21 + \frac{y}{3} - 2y$.
16. $\frac{R}{2} - \frac{R}{4} + \frac{R}{3} - \frac{5R}{12} + \frac{3R}{10} = 7$.
17. $\frac{x - 3}{7} + \frac{x + 5}{3} - \frac{x + 2}{6} = 4$.
18. $\frac{x+6}{x+5} - \frac{x+5}{x+4} = \frac{x+3}{x+2} - \frac{x+2}{x+1}$.
Hint: Simplify each member separately.

19. Solve for y: $1 - \frac{7}{ab} = \frac{ab}{y} - \frac{49}{aby}$.

20. Solve for r: $\frac{r - 1}{a - 1} - 1 = \frac{r^2 - a^2}{(a - 1)(a - 1)}$.

4. Formulas. In engineering, the relationships among quantities are shown by means of expressions written with letters, symbols, and constants. Such expressions are called *formulas* and are valuable aids to the engineer because they present a concise method of stating laws that pertain to scientific relationships. *A formula is simply an equation and all the methods for the solution of equations are applicable in dealing with formulas.*

Examples of formulas are numerous in the engineering field. For instance, the voltage E across any part of a direct-current electrical circuit is equal to the product of the current I through that part of the circuit and the resistance R of that part of the circuit, or in formula form, $E = IR$. In the same manner, the power P drawn by the circuit is equal to the product of the voltage E and the current I or in formula form, $P = EI$.

In many formulas, the symbol that is used to represent a quantity is the first letter of the name of the quantity. For instance, in the formula $HP = \frac{PLAN}{33{,}000}$, which is an expression for the horsepower of a steam engine, the symbols are the first letters of the names of the quantities. Thus,

HP = horsepower; P = steam pressure in lb/sq in. (psi);
L = piston stroke length in ft; A = piston head area in sq in.;
N = number of piston strokes per minute.

Check Original area = 14 ft by 10 ft = 140 sq ft,
new area = 12 ft by 11 ft = 132 sq ft.
140 sq ft − 8 sq ft = 132 sq ft.

6. Primes and subscripts. In the engineering field it is often desirable to make a distinction between several values of the same quantity. Thus two resistances may be used in the same formula and some method of differentiating between them must be found. Rather than using x and y, it has become common practice to represent such quantities by means of subscripts. For the two resistances mentioned, the symbols R_1 and R_2 or R_a and R_b would be used. The small number or letter written a little below and to the right of the R is called a *subscript.* R_1 and R_2 are read "R sub one" and "R sub two" or simply "R one" and "R two." R_a and R_b are read "R sub a" and "R sub b." This system of subscripts enlarges our field of symbols and makes them easier to read. The subscript is placed below and to the right of the symbol in order to distinguish it from an *exponent*, which is a small number placed a little above and to the right of the symbol. Thus I^2 means I times I, but I_2 means "I sub two" or "I two."

Primes and seconds are also used to distinguish between quantities. Thus one current may be denoted by I' and another by I''. The first one is read "I prime" and the second is read "I double prime" or "I second." This system is not as satisfactory as the system of subscripts because the prime marks are not always made as clear as numbers and are therefore more difficult to identify.

EXERCISE 4-2

Given	*Solve for*	*Given*	*Solve for*
1. $A = \pi R^2$	R^2	10. $S = V_0 t - \frac{1}{2} gt^2$	V_0
2. $A = \frac{1}{2}(a + b)h$	a	11. $(KE) = \frac{1}{2} mv^2$	m
3. $V = \frac{4}{3}\pi R^3$	R^3	12. $A = \frac{D}{L^2}(L^2 - a^2)$	L
4. $V = \frac{1}{3}\pi R^2 h$	R^2	13. $R_t = R_1 + R_2 + R_3$	R_3
5. $S = 2\pi Rh$	h	14. $\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$	R_t
6. $HP = \frac{2\pi NT}{33{,}000}$	T	15. $L = \frac{0.4\pi N^2 \mu A}{l10^8}$	μ
7. $C = \frac{5}{9}(F - 32)$	F	16. $x = 2\pi fl$	l
8. $I = P(1 + r)^n$	P	17. $x_c = \frac{10^6}{2\pi fc}$	c
9. $V = \frac{1}{6}\pi h(3R^2 + h^2)$	R^2	18. $e = 4NS\Phi\, 10^{-8}$	Φ

Given	Solve for
19. $s = K\dfrac{V - I_aR_a}{\Phi}$	V
20. $f = \dfrac{qq^1}{r^2}$	q
21. $\dfrac{I_p}{I_s} = \dfrac{N_s}{N_p}$	N_s
22. $N = \dfrac{120f}{p}$	f
23. $\Phi = \dfrac{0.4\pi NI}{\dfrac{l}{A\mu}}$	I

Given	Solve for
24. $P = RI^2$	R
25. $E = IR$	I
26. $e_b = E_b - iR_l$	R_l
27. $\dfrac{R_1}{R_2} = \dfrac{R_3}{R_4}$	R_4
28. $F = SKPBy$	S
29. $(BMEP) = \dfrac{150.8T}{D}$	T
30. $t = \dfrac{T(C - F)}{D}$	T
31. $N = IP + 2$	P

32. A wattmeter and an ammeter together cost \$77. The cost of the wattmeter is \$2 more than two times the cost of the ammeter. What is the cost of each?

33. A sum of \$2,500 is divided between A and B with A receiving \$1 for every \$4 that B receives. How much will each receive?

34. If two times a number is added to 16 the sum will be 80. What is the number?

35. If two times a number is added to 18 the sum will be equal to five times the number. What is the number?

36. Find three consecutive even numbers whose sum is 54.

37. Divide 130 into three parts so that the second shall be two times the first, and the third five times the second.

38. A steamer can run 20 mph in still water. It can run 48 miles with the current in the same time that it will run 32 miles against the current. What is the speed of the current?

39. The sum of two numbers is 45 and their difference increased by 3 is equal to the smaller number. What are the numbers?

40. A merchant after selling one sixth, one fourth, and one third of a piece of material had 24 yd remaining. How many yards were in the piece originally?

41. A man has \$9.25 in dimes and quarters, and he has five fewer quarters than he has dimes. How many of each has he?

42. One barrel contained 88 qt of wine and another 48 gal. From the second two times as much wine was drawn as from the first, and then the second contained three times as much wine as the first. How much did each contain?

43. A father has saved \$600 yearly for the past 50 yr. Each of his four sons has saved the same amount yearly: the first, for the past 28 yr; the second, for 25 yr; the third, for 18 yr; and the fourth, for 16 yr. How many years ago had the father saved an amount equal to the combined savings of his sons?

2 yrs

67. Equation 27 shows the relationship among the four resistances of a Wheatstone bridge.

(a) Determine R_4 when $R_1 = 20$, $R_2 = 50$, and $R_3 = 15$.
(b) Determine R_1 when $R_2 = 125$, $R_3 = 115$, and $R_4 = 150$.
(c) Determine R_2 when $R_1 = 1{,}000$, $R_3 = 15$, and $R_4 = 0.3$.
(d) Determine R_3 when $R_1 = 150$, $R_2 = 15$, and $R_4 = 20$.

68. Solve the following formula for t:

$$H = (1.6 \times 10^6) \frac{b_1 - b_2}{b_1 + b_2} (1 + 0.004t).$$

69. Solve the following equation for t_2, $R_2 = R_1[1 + \alpha_1(t_2 - t_1)]$. Determine t_2 when $R_1 = 43.5$, $R_2 = 48.5$, $t_1 = 20°$ C, and $\alpha_1 = 0.00393$.

7. Simultaneous equations.* In the equation $x - y = 3$, it is evident that numerous values may be substituted for the unknowns x and y to satisfy the equation. Thus $x = 4$ and $y = 1$ is one pair of values that will satisfy the equation. Other pairs of values are $x = 5$, $y = 2$; $x = 6$, $y = 3$; $x = 7$, $y = 4$, and so on indefinitely. Such an equation as this is called an *indeterminate equation*.

Now if we take two indeterminate equations, each will have an indefinite number of pairs of values to satisfy the equation but usually there will be only one pair of values that will satisfy both equations, provided both equations are of the first order. Thus in the equations

$$(1)\ x + y = 10$$

and

$$(2)\ x - y = 2$$

there are any number of pairs of values that will satisfy each one. In equation (1) pairs of values that satisfy are: $x = 9$, $y = 1$; $x = 8$, $y = 2$; $x = 6$, $y = 4$; $x = 5$, $y = 5$; $x = 4$, $y = 6$, etc.

In Equation (2) pairs of values that satisfy are $x = 4$, $y = 2$; $x = 5$, $y = 3$; $x = 6$, $y = 4$; $x = 8$, $y = 6$, etc.

It may be seen, however, by examining the pairs of values that satisfy each equation, that only one pair is common to both equations, this pair being $x = 6$ and $y = 4$.

Two equations that have only one pair of values that satisfy both are called *independent simultaneous equations,* and the process of solving such equations for the pair of values that fit both is called *solving simultaneously.*

If the equations are such that no pair of values will fit both, then they are termed *inconsistent equations.*

If the equations are such that every pair of values that satisfy one will satisfy the other, then they are termed *equivalent equations* or *identical equations.*

* The solution of simultaneous equations by graphical methods is discussed in Chapter 5, Part I.

It is evident that setting down numerous pairs of values for each equation and then picking the pair that will fit both equations is a long-drawn-out process and not at all convenient.

So, shorter processes have been devised for the solution of simultaneous equations. The various methods of solution are alike in that one of the unknowns is eliminated and a single equation found containing the other unknown, which then can be solved as a simple equation in one unknown. The following methods are in common use for solving two equations simultaneously:

1. Addition or subtraction.
2. Substitution.
3. Comparison.

8. Elimination by addition or subtraction. If the coefficients of either the x terms or the y terms are the same in the two equations, that unknown can be eliminated by the addition or the subtraction of the equations, depending upon the signs of the coefficients. If the signs are alike, the equations are subtracted; if the signs are unlike, the equations are added. If the coefficients of neither the x terms nor the y terms are the same, they can be made the same for one of the unknowns by multiplying each equation through by the proper constant.

Example 4–14. Solve the equations $x + y = 10$ and $x - y = 2$ for values of x and y.

Solution: Since the coefficients of the y terms are the same and their signs are unlike, the two equations can be added directly. Thus by addition,

$$\begin{array}{r} x + y = 10 \\ \underline{x - y = \ \ 2} \\ 2x = 12, \end{array}$$

whence

$$x = 6.$$

Also, since the coefficients of the x terms are the same and the signs are alike, the second equation can be subtracted from the first to solve for y. Thus by subtraction,

$$\begin{array}{r} x + y = \ \ 10 \\ +x - y = -2 \\ \underline{- \quad + \qquad\qquad} \\ 2y = \ \ \ 8, \end{array}$$

whence

$$y = 4.$$

Example 4–15. Solve the equations

$$4x + 3y = 24$$

and

$$5x - 2y = 7.$$

Solution: In order to eliminate one of the unknowns, the first step is to make the absolute values of the coefficients of one of the unknowns the same in both equations. If the first equation is multiplied by 2 and the

second equation by 3, the y term in each equation will have a coefficient of 6. Thus,

$$(2)(4x) + (2)(3y) = (2)(24)$$

and
$$(3)(5x) - (3)(2y) = (3)(7),$$

or
$$8x + 6y = 48,$$

and
$$15x - 6y = 21.$$

By addition,
$$23x \qquad = 69,$$

whence
$$x = 3.$$

To solve for y, the first equation can be multiplied by 5 and the second equation by 4 to make the coefficients of the x terms the same. Then the equations are subtracted. Thus,

$$(5)(4x) + (5)(3y) = (5)(24),$$
$$(4)(5x) - (4)(2y) = (4)(7),$$

or
$$20x + 15y = 120,$$
$$20x - 8y = 28.$$

Subtracting, we get

$$\begin{array}{rcrcr} +20x & + & 15y & = & +120, \\ +20x & - & 8y & = & +28, \\ \underline{-} & & \underline{+} & & \underline{-} \\ & & 23y & = & 92, \\ & & y & = & 4. \end{array}$$

In Examples 4–14 and 4–15 the second unknown could have been found by substitution of the first unknown into either of the original equations. Thus in Example 4–14 the substitution of 6, the value for x, in the first equation gives

$$6 + y = 10$$

or
$$y = 10 - 6$$

and
$$y = 4.$$

Also in Example 4–15 the substitution of 3 for x in the first equation gives

$$(4)(3) + 3y = 24$$

or
$$3y = 24 - 12,$$
$$3y = 12,$$

whence
$$y = 4.$$

This substitution of the value of one unknown in order to solve for the other unknown is satisfactory provided the work has been carefully done in solving for the first unknown so that its value is correct. However, the possibility of an error in solving for the value of this first unknown is always present and any error in its value will produce an error in the second unknown. Therefore, it is better generally to solve for each unknown independently as in Examples 4–14 and 4–15. If the coefficients are letters instead of numbers, the same method of solution is applicable.

EXAMPLE 4–16. Solve the equations

$$ax - by = p,$$
$$cx + dy = r.$$

Solution: Multiply the first equation by d and the second equation by b to make the y terms alike, and then add the equations

$$\begin{aligned} adx - bdy &= dp, \\ bcx + bdy &= br. \end{aligned}$$

Then by addition,

$$adx + bcx = dp + br$$

or

$$(ad + bc)x = dp + br,$$

whence

$$x = \frac{dp + br}{ad + bc}.$$

To solve for y, multiply the first equation by c and the second equation by a to make the x terms alike and then subtract the equations.

$$\begin{aligned} acx - bcy &= cp, \\ \underline{acx + ady} &\underline{= - ar.} \end{aligned}$$

By subtraction,

$$-ady - bcy = cp - ar.$$

Changing all signs,

$$ady + bcy = ar - cp.$$

Then

$$(ad + bc)y = ar - cp,$$

whence

$$y = \frac{ar - cp}{ad + bc}.$$

9. Elimination by substitution. In this method, one equation is solved for one unknown in terms of the other and this value substituted in the second equation.

EXAMPLE 4–17. Solve the equations

$$\begin{aligned} 6x + y &= 33, \\ x - y &= 2. \end{aligned}$$

Solution: From the second equation: $x = 2 + y$.

By substituting this value for x in the first equation we get

$$(6)(2 + y) + y = 33.$$

Then

$$\begin{aligned} 12 + 6y + y &= 33, \\ 7y &= 33 - 12 = 21, \end{aligned}$$

whence

$$y = 3.$$

Also, from the second equation, $y = x - 2$.

By substituting this value for y in the first equation we get

$$\begin{aligned} 6x + (x - 2) &= 33, \\ 6x + x &= 33 + 2, \\ 7x &= 35, \\ x &= 5. \end{aligned}$$

10. Elimination by comparison. Each equation may be solved for one of the unknowns in terms of the other and these results equated to solve for the other unknown.

EXAMPLE 4–18. Solve the equations

$$\begin{aligned} 2x + 3y &= 39, \\ 3x - 4y &= 16. \end{aligned}$$

To check these values, they should be substituted in the other two original equations.

$$\text{Equation (2): } (2)(2) - 3 - 4 = -3.$$
$$\text{Equation (3): } 2 - 3 + (2)(4) = 7.$$

The method of solving a system of linear equations in three unknowns as suggested in Example 4–20 is as follows:

1. Determine by inspection the unknown to be eliminated first.

2. Eliminate this unknown from each of two pairs of equations. This will leave two equations with two unknowns.

3. Solve these two equations by one of the methods for the solution of linear equations in two unknowns.

4. Substitute these two values in one of the original equations to get the third unknown.

5. Check the results by substituting the values found in the other original equations.

13. Kirchhoff's laws. A practical application of the solution of simultaneous equations is found in Kirchhoff's two laws for the electric circuit. These two laws are as follows:

1. *The algebraic sum of the currents about any point in an electrical network is zero.* This can be restated to read that the sum of the currents entering a point is equal to the sum of the currents leaving the point.

2. *The algebraic sum of all the electromotive forces and voltage drops around any closed electrical path is zero.* This can be restated to read that the algebraic sum of the electromotive forces is equal to the algebraic sum of the resistance voltage drops around a closed circuit.

The use of Kirchhoff's laws in an electrical network leads to a number of simultaneous equations to be solved, the number depending upon the number of unknown quantities. The unknown quantities which consist

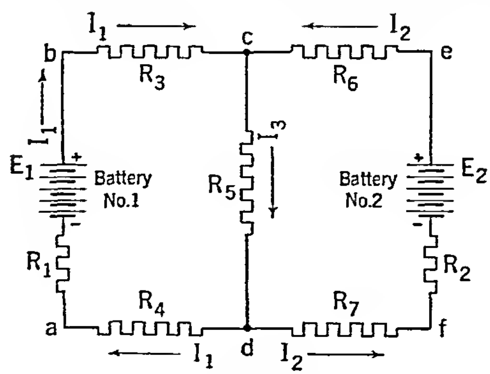


Fig. 4–1.

of voltages, currents, and resistances are denoted by the letters E, I, and R respectively with subscripts. A typical case is shown in Fig. 4–1. (*Note:* R_1

represents the resistance of battery 1. R_2 represents the resistance of battery 2.)

By Kirchhoff's first law, the current equation at point C is

$$I_1 + I_2 = I_3.$$

By Kirchhoff's second law, two voltage drop equations can be set up, one for the path *abcda* and the second for path *ecdfe*.

$$E_1 = I_1R_1 + I_1R_3 + I_3R_5 + I_1R_4;$$
$$E_2 = I_2R_2 + I_2R_6 + I_3R_5 + I_2R_7.$$

Numerical values of voltages and resistances are substituted and these three equations solved simultaneously.

EXAMPLE 4–21. Solve the above three simultaneous equations when $E_1 = 12$ v, $E_2 = 10$ v, $R_1 = 0.5$ ohm, $R_2 = 0.4$ ohm, $R_3 = 4$ ohms, $R_4 = 6$ ohms, $R_5 = 2$ ohms, $R_6 = 5$ ohms, and $R_7 = 3$ ohms.

Solution: By substituting these values in the two voltage equations, there results:

$$12 = 0.5I_1 + 4I_1 + 2I_3 + 6I_1;$$
$$10 = 0.4I_2 + 5I_2 + 2I_3 + 3I_2.$$

Combining like terms we have

$$10.5I_1 + 2I_3 = 12;$$
$$8.4I_2 + 2I_3 = 10.$$

These two equations cannot be solved simultaneously because there are three unknowns in them, but by combining them with the current equation we shall have three independent equations with three unknowns and, therefore, we can solve for the three currents I_1, I_2, and I_3. Thus, we now have the three equations

$$10.5I_1 + 2I_3 = 12;$$
$$8.4I_2 + 2I_3 = 10;$$
$$I_1 + I_2 = I_3.$$

By substituting the value of I_3 from the third equation in each of the first two equations we shall eliminate I_3 and have two equations left with I_1 and I_2.

$$10.5I_1 + 2(I_1 + I_2) = 12;$$
$$8.4I_2 + 2(I_1 + I_2) = 10.$$

Simplifying,

$$12.5I_1 + 2I_2 = 12;$$
$$2I_1 + 10.4I_2 = 10.$$

Multiplying the first by 10.4 and the second by 2 and subtracting we shall eliminate I_2.

$$(10.4)(21.5I_1) + (10.4)(2I_2) = (10.4)(12),$$
$$(2)(2I_1) + (2)(10.4I_2) = (2)(10).$$

Simplifying,

$$\begin{array}{r} 130I_1 + 20.8I_2 = 124.8 \\ \underline{\begin{matrix} +4I_1 + 20.8I_2 = +20 \\ - \quad\;\; - \qquad\quad - \end{matrix}} \\ 126I_1 \qquad\qquad = 104.8 \end{array}$$

$$I_1 = \frac{104.8}{126} = 0.832 \text{ amp.}$$

Substituting this value in the equation $2I_1 + 10.4I_2 = 10$, we get

$$(2)(0.832) + 10.4I_2 = 10,$$
$$10.4I_2 = 10 - 1.664,$$
$$I_2 = \frac{8.336}{10.4} = 0.802 \text{ amp.}$$

Then $$I_3 = I_1 + I_2$$
or $$I_3 = 0.832 + 0.802 = 1.634 \text{ amp.}$$

These answers can be checked by substituting them in the original equations.

EXERCISE 4-3

Solve the following sets of simultaneous equations for all the unknown values:

1. $2x - 3y = 8$,
   $3x + 2y = 25$.
2. $7x - 5y = 41$,
   $3x - y = 21$.
3. $5x - 3y = 7$,
   $x + 2y = 17$.
4. $3b - a = 19$,
   $3b + a = 29$.
5. $2y - 3x = 5$,
   $4x + y = 19$.
6. $5m - 4n = 41$,
   $2m + n = 19$.
7. $x + 6 = y - 6$,
   $2(x + 2) = 22 - y$.
8. $(x + 2) + (y - 1) = 16$,
   $(x - 1) - (y - 4) = 4$.
9. $\dfrac{x+y}{3} + \dfrac{x-y}{4} = 10$,
   $\dfrac{x+y}{2} - \dfrac{x-y}{8} = 11$.
10. $\dfrac{a+b}{5} - \dfrac{b-a}{5} = \dfrac{5a-b}{8} + \dfrac{2(b-a)-70}{5} + 1$,
    $6a = 2b$.
11. $8I_1 + 5I_2 = 44$,
    $7I_1 - 3I_2 = 9$.
12. $35a + 17b = 86$,
    $56a - 13b = 17$.
13. $0.5I_1 + 0.7I_2 = 1.1$,
    $0.14I_1 - 0.06I_2 = 4.42$.
14. $cx + dy = c^2 + 2c + d^2$,
    $dx + cy = c^2 + 2d + d^2$.
    (*Hint:* Subtract the two equations and cancel the common factor as the first step.)
15. Solve for x and y in terms of m and n.

$$\frac{x}{n+m} + \frac{y}{n-m} = 2,$$
$$nx - my = n^2 + m^2.$$

16. $\dfrac{3y + 2x}{5} + \dfrac{y + 6}{7} = 2,$
$\dfrac{2x - 5y}{3} + \dfrac{x + 7}{4} = 1.$

17. $a + b + c - d = 2,$
$a + b - c + d = 4,$
$a - b + c + d = 6,$
$-a + b + c + d = 8.$

18. $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 6,$
$\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{d} = 4,$
$\dfrac{1}{a} + \dfrac{1}{c} + \dfrac{1}{d} = 3,$
$\dfrac{1}{b} + \dfrac{1}{c} + \dfrac{1}{d} = 2.$

(*Hint:* Divide the sum of the four equations by 3 to obtain an equation from which to subtract each of the given equations.)

19. $\dfrac{a}{4} + \dfrac{b}{5} + \dfrac{c}{6} = 76,$
$\dfrac{a}{3} + \dfrac{b}{4} + \dfrac{c}{5} = 94,$
$\dfrac{a}{2} + \dfrac{b}{3} + \dfrac{c}{4} = 124.$

20. $7x - 4y + z = 12,$
$5x - y + z = 48,$
$x + 3y - 4z = 40.$

21. $x + y + z = 10,$
$u + y + z = 14,$
$u + x + z = 13,$
$u + x + y = 11.$
(See *Hint* above.)

22. $3u + x + 2y - z = 22,$
$4u + 3x - 2y = 19,$
$u + 2y + z = 23,$
$4x - y + 3z = 35.$

23. Solve the three simultaneous equations of Fig. 4-1 if $E_1 = 24$ v, $E_2 = 16$ v, $R_1 = 0.2$ ohm, $R_2 = 0.1$ ohm, $R_3 = 5.5$ ohms, $R_4 = 3.6$ ohms, $R_5 = 2.5$ ohms, $R_6 = 4.8$ ohms, and $R_7 = 1.75$ ohms.

24. Find two numbers whose sum is 25 and whose difference is 7.

25. If one number is multiplied by 3 and another by 7, the sum of the products will be 45; if the first is multiplied by 7 and the second by 3, the sum of the products will be 65. What are the numbers?

26. If 1 is subtracted from the numerator and also from the denominator of a certain fraction, the result will be $\frac{1}{2}$ but if 1 is added to the numerator and also to the denominator of the fraction, the result will be $\frac{3}{5}$. What is the fraction?

27. A and B make a purchase for \$64. A uses all of his money, and B three fourths of his. If A had used three fourths of his money and B all of his, they would have paid \$2 less. How much money had each?

28. A mechanic and an apprentice together receive \$104. The mechanic works 7 days and the apprentice 12 days; and the mechanic earns in 3 days \$4 more than the apprentice earns in 5 days. What wages will each receive per day?

29. A tank has two pumps. If the first is used 2 hr and the second 3 hr, 1,100 cu ft of water will be discharged. But if the first is used 1 hr and the second $2\frac{1}{2}$ hr, 750 cu ft of water will be discharged. How many cubic feet of water can each pump discharge in 1 hr?

30. A and B jointly contribute \$20,000 to a business. A leaves his money in the business 18 months, and B leaves his money 2 yr and 8 months. If their profits are equal, how much does each contribute?

31. In a factory in which 500 men and women are employed the average weekly pay for men is \$48 and for women \$32. If the weekly payroll is \$22,000, find the number of men and the number of women employed.

32. Solve the equations $E_e = 0.707E_m$ and $E_a = 0.636E_m$ simultaneously for E_a in terms of E_e. If $E_e = 120$ v, what is the value for E_a?

33. Solve the following equations for E and V:

$$I_1E + I_2V = I_2r + I_1R;$$
$$I_2E + I_1V = I_1r + I_2R.$$

34. Two passenger planes start from Chicago at the same time, one traveling east and the other traveling west. The one traveling east has a speed of 60 mph faster than the one traveling west. At the end of 4 hr, they are 1,800 miles apart. What is the speed of each?

35. In measuring the resistance of the coils of an a-c generator, the following equation is used:

$$\frac{1}{R_0} = \frac{1}{R} + \frac{1}{2R}$$

where R_0 is the resistance measured between any two terminals and R is the resistance per coil. Solve this equation for R and determine the value of R when $R_0 = 0.16$ ohm.

36. The circumference of the rear wheel of a tractor exceeds the circumference of the front wheel by 4 ft and the front wheel makes the same number of revolutions in running 200 yd that the rear wheel will make in running 250 yd. What is the diameter of each wheel?

37. The perimeter of a rectangular lawn, with a path of uniform width around it, is 420 ft. The total area of the lawn and path together is larger than two times the difference of their areas by 1,200 sq yd, and the width of the path is one sixth the size of the shorter side of the lawn. Find the dimensions of lawn and path.

38. A merchant received from one customer \$33 for 12 yd of silk and 6 yd of cloth; and from another customer \$42 for 15 yd of silk and 8 yd of cloth at the same prices. What was the price of the silk and of the cloth?

39. A man walks 27 miles, first at the rate of 4 mph, and later at the rate of 3 mph. If he had walked 4 mph when he walked 3 and 3 mph when he walked 4, he would have gone 2 miles farther. How far would he have gone if he had walked 4 mph the whole time?

40. If the base of a rectangle is increased by 3 ft and the altitude is decreased by 2 ft, the area will be increased by 14 sq ft. If the base is increased by 2 ft and the altitude is decreased by 3 ft, the area will be decreased by 46 sq ft. Find the base and the altitude of the rectangle.

41. The report of a cannon travels with the wind 344.4 yd a second, and against the wind 335.9 yd a second. What is the velocity of the report in still air, and what is the velocity of the wind in mph?

42. A and B formed a partnership. A invested \$15,000 of his money and \$10,000 that he borrowed; B invested \$23,000 of his own money and

\$7,000 that he borrowed at the same rate of interest as was paid by A. At the end of a year A's share in the profits amounted to \$1,500 more than the interest on his \$10,000 and B's share amounted to \$2,050 more than the interest on his \$7,000. What rate of interest did they pay, and what rate of interest did they realize on their investments?

43. Two bodies move along the circumference of a circle in the same direction from two different points, the shorter distance between which, measured along the circumference, is 180 ft. One body will overtake the other in 18 sec if they move in one direction, or in 24 sec if they move in the opposite direction. While the one goes once around the circumference, the other will cover 105 ft more than the circumference. Determine circumference of the circle, and the rate at which each body moves.

44. An alloy of tin and lead, weighing 40 lb, loses 4 lb in weight when immersed in water. Find the amount of tin and lead in the alloy if 10 lb of tin loses 1.4 lb when immersed in water, and 5 lb of lead loses 0.375 lb.

45. Four men are to do a piece of work. A and B can do the work in 20 days, A and C in 18 days, A and D in 15 days, and B, C, and D in $11\frac{1}{4}$ days. In how many days can each man do the work, and in how many days can they do the work if they work together?

46. In the two formulas, $s = \frac{1}{2}at^2$ and $V = at$, find V in terms of s and a by eliminating t from the two formulas.

14. Determinants. A method of solving simultaneous linear equations that is in common use, particularly among authors of higher mathematics texts and engineering texts, is known as *determinant notation. A determinant is a square of numbers arranged in rows and columns and enclosed by two vertical lines.* If there are two rows and two columns, the determinant is one of the *second order;* if three rows and three columns, it is one of the *third order,* and so on. The numbers in the determinant are called its *elements.* Thus the expression

$$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}$$

is a determinant of the second order, and

$$\begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix}$$

is a determinant of the third order. We shall show methods of using determinants for the solution of two and three simultaneous linear equations with two and three unknowns respectively.

15. Determinants of the second order. A general formula can be developed for the solution of any two simultaneous equations.

The two general equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ can be solved for x and y in terms of the coefficients and the constant terms by the method given under addition and subtraction. Multiplying the

first equation by b_2 and the second equation by b_1 and subtracting the second from the first gives

$$\begin{array}{rl} a_1b_2x + b_1b_2y = & b_2c_1 \\ \underset{-}{+}\, a_2b_1x \underset{-}{+}\, b_1b_2y = & \underset{-}{+}\, b_1c_2 \\ \hline a_1b_2x - a_2b_1x = & b_2c_1 - b_1c_2. \end{array}$$

Solving for x, we get

$$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2$$

or

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}.$$

Again, multiplying the first equation by a_2 and the second equation by a_1 and subtracting the first from the second gives

$$\begin{array}{rl} a_1a_2x + a_1b_2y = & a_1c_2 \\ \underset{-}{+}\, a_1a_2x \underset{-}{+}\, a_2b_1y = & \underset{-}{+}\, a_2c_1 \\ \hline a_1b_2y - a_2b_1y = & a_1c_2 - a_2c_1. \end{array}$$

Solving for y, we get

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1$$

or

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

These values for x and y can be used as formulas for solving any simultaneous linear equations in two unknowns. The coefficients a_1, a_2, b_1, and b_2 and the constant terms c_1 and c_2 may have any numerical values positive or negative, or zero, provided $a_1b_2 - a_2b_1$ does not equal zero.

The symbol
$$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}$$

is used for the expression $a_1b_2 - a_2b_1$. It is made up of two columns and two rows of numbers with a vertical line on either side and is a determinant of the second order. Each number in a column or row is an *element*, and the expression $a_1b_2 - a_2b_1$ is the *expansion of the determinant.*

The value of the determinant of the second order can be found by subtracting the product of the numbers in the diagonal from the lower left upward from the product of the numbers in the diagonal from the upper left downward. Thus:

$$\begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix} = (4)(5) - (3)(2) = 20 - 6 = 14.$$

Evidently, the values for x and y that we have found can be expressed in this determinant notation because the numerator can be put in this same form just as well as the denominator. Thus,

$$b_2c_1 - b_1c_2 = \begin{vmatrix} c_1b_1 \\ c_2b_2 \end{vmatrix} \quad \text{and} \quad a_1c_2 - a_2c_1 = \begin{vmatrix} a_1c_1 \\ a_2c_2 \end{vmatrix}.$$

Therefore,
$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

The denominators in these two expressions are the same determinant, which is made up of the coefficients of x and y. It is called the *determinant of the system.* The numerator of the expression for x differs from the denominator in having the constants c_1 and c_2 in the first column in place of the coefficients a_1 and a_2. The numerator of the expression for y differs from the denominator in having the constants c_1 and c_2 in the second column in place of the coefficients b_1 and b_2.

By the use of these formulas we are able to write down at once the solution for the unknowns in any system of independent linear equations containing only two unknowns and expressed in the general form:

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2.$$

EXAMPLE 4–22. Solve the following simultaneous equations by determinants:

$$2x - 5y = -25;$$
$$3x + y = 22.$$

Solution: In these equations $a_1 = 2$; $b_1 = -5$; $c_1 = -25$; $a_2 = 3$; $b_2 = 1$; $c_2 = 22$.

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} -25 & -5 \\ 22 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & 1 \end{vmatrix}} = \frac{(-25)(1) - (22)(-5)}{(2)(1) - (3)(-5)} = \frac{-25 + 110}{2 + 15} = \frac{85}{17} = 5;$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} 2 & -25 \\ 3 & 22 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & 1 \end{vmatrix}} = \frac{(2)(22) - (3)(-25)}{(2)(1) - (3)(-5)} = \frac{44 + 75}{2 + 15} = \frac{119}{17} = 7.$$

It should be noted that the signs of each of the coefficients is considered in the solution.

16. Determinants of the third order. The solution of three simultaneous equations with three unknown values can also be performed by determinant notation. For example, consider the three typical equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3.$$

Elimination of z from the first two equations is done by multiplying the first equation by c_2 and the second equation by c_1 and subtracting the second from the first. This results in the following equation with the unknowns x and y:

$$(a_1c_2 - a_2c_1)x + (b_1c_2 - b_2c_1)y = c_2d_1 - c_1d_2.$$

Elimination of z from the last two equations is accomplished by multiplying the second by c_3 and the third by c_2 and then subtracting the third from the second. This results in the following equation with the unknowns x and y:

$$(a_2c_3 - a_3c_2)x + (b_2c_3 - b_3c_2)y = c_3d_2 - c_2d_3.$$

One of the unknowns is now eliminated from these last two equations to give a single equation in the other unknown. Let us eliminate y by multiplying each equation by the coefficient of y in the other and subtracting the equations. Thus we get

$$\begin{array}{rl}
(a_1c_2-a_2c_1)(b_2c_3-b_3c_2)x+(b_1c_2-b_2c_1)(b_2c_3-b_3c_2)y & =(b_2c_3-b_3c_2)(c_2d_1-c_1d_2) \\
+(a_2c_3-a_3c_2)(b_1c_2-b_2c_1)x+(b_1c_2-b_2c_1)(b_2c_3-b_3c_2)y & =+(b_1c_2-b_2c_1)(c_3d_2-c_2d_3) \\
- \qquad\qquad\qquad\qquad - \qquad\qquad\qquad & \;\; - \\
\hline
[(a_1c_2-a_2c_1)(b_2c_3-b_3c_2)-(a_2c_3-a_3c_2)(b_1c_2-b_2c_1)]x & =(b_2c_3-b_3c_2)(c_2d_1-c_1d_2) \\
 & \quad -(b_1c_2-b_2c_1)(c_3d_2-c_2d_3).
\end{array}$$

By performing the indicated multiplications and solving for x, there results

$$x = \frac{b_2c_2c_3d_1-b_3c_2^2d_1-b_2c_1c_3d_2+b_3c_1c_2d_2-b_1c_2c_3d_2+b_2c_1c_3d_2+b_1c_2^2d_3-b_2c_1c_2d_3}{a_1b_2c_2c_3-a_1b_3c_2^2-a_2b_2c_1c_3+a_2b_3c_1c_2-a_2b_1c_2c_3+a_2b_2c_1c_3+a_3b_1c_2^2-a_3b_2c_1c_2}.$$

It will be found upon inspection that the third and sixth terms of the numerator can be canceled since they are the same and have opposite signs. Also the third and sixth terms of the denominator can be canceled for the same reason. Also c_2 is present in each remaining term of both numerator and denominator. Therefore, we can simplify the expression by canceling out the like terms, dividing both numerator and denominator by c_2, and rearranging the terms to put all the positive ones together and all the negative ones together thus:

$$x = \frac{d_1b_2c_3 + b_1c_2d_3 + c_1d_2b_3 - d_3b_2c_1 - b_3c_2d_1 - c_3d_2b_1}{a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1}.$$

In like manner y and z may be obtained:

$$y = \frac{a_1d_2c_3 + d_1c_2a_3 + c_1a_2d_3 - a_3d_2c_1 - d_3c_2a_1 - c_3a_2d_1}{a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1},$$

$$z = \frac{a_1b_2d_3 + b_1d_2a_3 + d_1a_2b_3 - a_3b_2d_1 - b_3d_2a_1 - d_3a_2b_1}{a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1}.$$

It will be observed that the denominators are identical and that each consists of three positive terms and three negative terms. Such an expression is in the form of the expansion of a determinant and can be represented by the determinant

$$\begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix}.$$

This is a determinant of the third order because it has three columns and three rows and it is called the determinant of the system because it is formed from the coefficients of the unknowns.

There is an easy way of remembering how to expand a determinant of the third order.

(1) Write the determinant and then write its first and second columns again at its right as follows:

$$\left|\begin{array}{c|c} a_1b_1c_1 & a_1b_1 \\ a_2b_2c_2 & a_2b_2 \\ a_3b_3c_3 & a_3b_3 \end{array}\right|.$$

(2) Write the product of the numbers in each diagonal running from the upper left downward and make each product positive, thus:

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3.$$

(3) Write the product of the numbers in each diagonal running from the lower left upward and make each product negative, thus:

$$-a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1.$$

(4) Combine these six terms to form the expansion of the determinant. *It must be remembered that this expansion applies to a determinant of the third order only and not to one of any higher order.*

The numerators of the expressions for x, y, and z can also be treated in the same way as the denominator since each is in the form of the expansion of a determinant. Therefore the complete determinant form for each of the three unknowns in its final form is as follows:

$$x = \frac{\left|\begin{array}{c|c} d_1b_1c_1 & d_1b_1 \\ d_2b_2c_2 & d_2b_2 \\ d_3b_3c_3 & d_3b_3 \end{array}\right|}{\left|\begin{array}{c|c} a_1b_1c_1 & a_1b_1 \\ a_2b_2c_2 & a_2b_2 \\ a_3b_3c_3 & a_3b_3 \end{array}\right|} \qquad y = \frac{\left|\begin{array}{c|c} a_1d_1c_1 & a_1d_1 \\ a_2d_2c_2 & a_2d_2 \\ a_3d_3c_3 & a_3d_3 \end{array}\right|}{\left|\begin{array}{c|c} a_1b_1c_1 & a_1b_1 \\ a_2b_2c_2 & a_2b_2 \\ a_3b_3c_3 & a_3b_3 \end{array}\right|} \qquad z = \frac{\left|\begin{array}{c|c} a_1b_1d_1 & a_1b_1 \\ a_2b_2d_2 & a_2b_2 \\ a_3b_3d_3 & a_3b_3 \end{array}\right|}{\left|\begin{array}{c|c} a_1b_1c_1 & a_1b_1 \\ a_2b_2c_2 & a_2b_2 \\ a_3b_3c_3 & a_3b_3 \end{array}\right|}$$

The products to be taken are indicated by straight lines drawn through the letters. *The numerator for each unknown has been obtained from the determinant of the system by replacing the coefficients of that unknown by the constant terms.* Thus the numerator for x has been formed from the determinant of the system by replacing the coefficients a_1, a_2, and a_3 by d_1, d_2, and d_3. Likewise the numerator for y has been formed from the determinant of the system by replacing the coefficients b_1, b_2, and b_3 by d_1, d_2, and d_3, and for z, c_1, c_2, and c_3 have been replaced by d_1, d_2, and d_3.

Example 4–23. Solve the following equations of Example 4–21 by determinants:

$$\begin{aligned} 10.5I_1 + 2I_3 &= 12; \\ 8.4I_2 + 2I_3 &= 10; \\ I_1 + I_2 - I_3 &= 0. \end{aligned}$$

Solution: The unknowns are I_1, I_2, and I_3, representing the x, y, and z respectively in the foregoing explanation. In the first equation, I_2 does

not appear and in the second equation I_1 does not appear. Therefore the coefficients for each will be zero. The equations could be written thus:

$$
\begin{aligned}
10.5I_1 + 0I_2 + 2I_3 &= 12;\\
0I_1 + 8.4I_2 + 2I_3 &= 10;\\
I_1 + I_2 - I_3 &= 0;
\end{aligned}
$$

$$
I_1 = \frac{\left|\begin{array}{ccc|cc} 12 & 0 & 2 & 12 & 0\\ 10 & 8.4 & 2 & 10 & 8.4\\ 0 & 1 & -1 & 0 & 1 \end{array}\right|}{\left|\begin{array}{ccc|cc} 10.5 & 0 & 2 & 10.5 & 0\\ 0 & 8.4 & 2 & 0 & 8.4\\ 1 & 1 & -1 & 1 & 1 \end{array}\right|} = \frac{\begin{array}{l}(12)(8.4)(-1) + (0)(2)(0) + (2)(10)(1)\\ \quad - (0)(8.4)(2) - (1)(2)(12)\\ \quad - (-1)(10)(0)\end{array}}{\begin{array}{l}(10.5)(8.4)(-1) + (0)(2)(1)\\ \quad + (2)(0)(1) - (1)(8.4)(2)\\ \quad - (1)(2)(10.5) - (-1)(0)(0)\end{array}}
$$

$$
I_1 = \frac{-100.8 + 20 - 24}{-88.2 - 16.8 - 21} = \frac{-104.8}{-126} = 0.832 \text{ amp.}
$$

$$
I_2 = \frac{\left|\begin{array}{ccc|cc} 10.5 & 12 & 2 & 10.5 & 12\\ 0 & 10 & 2 & 0 & 10\\ 1 & 0 & -1 & 1 & 0 \end{array}\right|}{\left|\begin{array}{ccc|cc} 10.5 & 0 & 2 & 10.5 & 0\\ 0 & 8.4 & 2 & 0 & 8.4\\ 1 & 1 & -1 & 1 & 1 \end{array}\right|} = \frac{\begin{array}{l}(10.5)(10)(-1) + (12)(2)(1)\\ \quad + (2)(0)(0) - (1)(10)(2)\\ \quad - (0)(2)(10.5) - (-1)(0)(12)\end{array}}{\begin{array}{l}(10.5)(8.4)(-1) + (0)(2)(1)\\ \quad + (2)(0)(1) - (1)(8.4)(2)\\ \quad - (1)(2)(10.5) - (-1)(0)(0)\end{array}}
$$

$$
I_2 = \frac{-105 + 24 - 20}{-126} = \frac{-101}{-126} = 0.802 \text{ amp.}
$$

$$
I_3 = \frac{\left|\begin{array}{ccc|cc} 10.5 & 0 & 12 & 10.5 & 0\\ 0 & 8.4 & 10 & 0 & 8.4\\ 1 & 1 & 0 & 1 & 1 \end{array}\right|}{\left|\begin{array}{ccc|cc} 10.5 & 0 & 2 & 10.5 & 0\\ 0 & 8.4 & 2 & 0 & 8.4\\ 1 & 1 & -1 & 1 & 1 \end{array}\right|} = \frac{\begin{array}{l}(10.5)(8.4)(0) + (0)(10)(1)\\ \quad + (0)(10)(1) - (1)(8.4)(12)\\ \quad - (1)(10)(10.5) - (0)(0)(0)\end{array}}{\begin{array}{l}(10.5)(8.4)(-1) + (0)(2)(1)\\ \quad + (2)(0)(1) - (1)(8.4)(2)\\ \quad - (1)(2)(10.5) - (-1)(0)(0)\end{array}}
$$

$$
I_3 = \frac{-100.8 - 105}{-126} = \frac{-205.8}{-126} = 1.634 \text{ amp.}
$$

17. Solution of third order determinants by minors. In Article 16 of this chapter it was shown that the third order determinant $\begin{vmatrix} a_1b_1c_1\\ a_2b_2c_2\\ a_3b_3c_3 \end{vmatrix}$ was equal to $a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$. It will be found that this expression contains a_1 in two terms, a_2 in two terms and a_3 in two terms. Therefore, by rearranging terms we get: $a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$. This we may also write in factor form as $a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$. Each of these terms is the product of an element and a second order determinant and can now be written in determinant form thus:

$$
a_1\begin{vmatrix} b_2c_2\\ b_3c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1c_1\\ b_3c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1c_1\\ b_2c_2 \end{vmatrix}
$$

Examination of the above will show that each product is formed by an element and a second order determinant which is obtained from the original third order determinant by *delcting the row and column in which the multiplying element is found.* Such a determinant thus formed is known as a *minor* of the element. Thus, the minor of the element a_1 is $\begin{vmatrix} b_2c_2 \\ b_3c_3 \end{vmatrix}$; the minor of the element a_2 is $\begin{vmatrix} b_1c_1 \\ b_3c_3 \end{vmatrix}$; and the minor of the element a_3 is $\begin{vmatrix} b_1c_1 \\ b_2c_2 \end{vmatrix}$. The total expression of elements and minors is an expansion of the third order determinant by minors.

We have expanded the third order determinant by minors of elements a_1, a_2, and a_3 in the first column. However, we could just as easily have performed the expansion by minors of the elements b_1, b_2, and b_3 in the second column or by minors of the elements c_1, c_2, and c_3 in the third column because the expanded expression contains each of the elements in two terms: b_1, b_2, and b_3 are each in two terms of the expanded expression, and a similar condition exists for c_1, c_2, and c_3.

Thus the expansion by minors of the elements b_1, b_2, and b_3 would give

$$-b_1\begin{vmatrix} a_2c_2 \\ a_3c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1c_1 \\ a_3c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1c_1 \\ a_2c_2 \end{vmatrix}$$

and the expansion by minors of the elements c_1, c_2, and c_3 would give

$$c_1\begin{vmatrix} a_2b_2 \\ a_3b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1b_1 \\ a_3b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix}$$

It is evident from an examination of these expansions that there is a definite significance in the sign preceding each element and its minor. There are two methods for determining this sign.

One method is to *add together the number of the column and the number of the row in which the element appears* in the original determinant. *If this sum is even, the sign of the term is positive; if the sum is odd, the sign is negative.*

The other method is to *move the element step by step to the prime or upper left hand corner* in the original determinant. *If the number of moves required is an even number, the sign is positive; if the number of moves required is an odd number, the sign is negative. If the element is already in the upper left hand corner, the sign is positive.*

Whichever method is used, *an even result gives a positive sign* and *an odd result gives a negative sign.*

To illustrate, let us examine the expansion by minors of the elements a_1, a_2, and a_3. Since a_1 is in the *first column* and the *first row,* the sign of its term will be positive because $1 + 1 = 2$, an *even* number. a_2 is in the *first column* and the *second row;* therefore, the sign of its term will be negative because $1 + 2 = 3$, an *odd* number. a_3 is in the *first column* and the

third row; therefore, the sign of its term will be positive because $1 + 3 = 4$, an *even* number. By the second method, a_1 is already in the upper left hand corner and therefore the term takes a positive sign; a_2 requires one move to reach this corner, therefore its term sign is negative; and a_3 requires *two* moves to reach this corner, therefore its term sign is positive.

When expanding by minors of the elements b_1, b_2, and b_3, we find that

	column and row	*moves*
b_1 is negative	$2 + 1 = 3$ (odd)	one move,
b_2 is positive	$2 + 2 = 4$ (even)	two moves,
b_3 is negative	$2 + 3 = 5$ (odd)	three moves.

The expansion of the third order determinant by minors will give the same answer regardless of the element chosen. Thus, expanding by minors of the elements a_1, a_2, and a_3 will give the same answer as expanding by minors of the elements b_1, b_2, and b_3 or the elements c_1, c_2, and c_3.

After a third order determinant has been expanded, the final solution is made simpler because it involves only the product of elements and second order determinants.

EXAMPLE 4–24. Solve the following set of simultaneous equations by minors:

$$x + y + z = 4;$$
$$2x - y + 3z = 14;$$
$$3x + 2y - 2z = -2.$$

Solution: The third order determinant of the coefficients is

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -2 \end{vmatrix}$$

Then

$$x = \frac{\begin{vmatrix} 4 & 1 & 1 \\ 14 & -1 & 3 \\ -2 & 2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -2 \end{vmatrix}}$$

Expanding both numerator and denominator by minors of the first column elements:

$$x = \frac{4\begin{vmatrix} -1 & 3 \\ 2 & -2 \end{vmatrix} - 14\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}}{1\begin{vmatrix} -1 & 3 \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}}$$

$$= \frac{4(2 - 6) - 14(-2 - 2) - 2(3 + 1)}{1(2 - 6) - 2(-2 - 2) + 3(3 + 1)}.$$

$$x = \frac{-16 + 56 - 8}{-4 + 8 + 12} = \frac{32}{16} = 2.$$

Likewise:

$$y = \frac{\begin{vmatrix} 1 & 4 & 1 \\ 2 & 14 & 3 \\ 3 & -2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -2 \end{vmatrix}}.$$

Since the denominator has not been changed, its expansion by minors will give the same result as for x, regardless of the elements used, and therefore will have the value of 16. The numerator may be expanded by using any row of elements desired. Let us use the first row.

$$y = \frac{1\begin{vmatrix} 14 & 3 \\ -2 & -2 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ -2 & -2 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 14 & 3 \end{vmatrix}}{16}$$

$$= \frac{(1)(-28 + 6) - 2(-8 + 2) + 3(12 - 14)}{16}$$

$$y = \frac{-22 + 12 - 6}{16} = \frac{-16}{16} = -1.$$

and

$$z = \frac{\begin{vmatrix} 1 & 1 & 4 \\ 2 & -1 & 14 \\ 3 & 2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -2 \end{vmatrix}}.$$

$$z = \frac{1\begin{vmatrix} -1 & 14 \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ 2 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ -1 & 14 \end{vmatrix}}{16}$$

$$= \frac{1(2 - 28) - 2(-2 - 8) + 3(14 + 4)}{16},$$

$$z = \frac{-26 + 20 + 54}{16} = \frac{48}{16} = 3.$$

18. Properties and transformation of determinants. The evaluation of a determinant of the third or higher order is simplified greatly when a number of zeros appear in a column or row. By choosing the elements of the column or row in which the zeros occur and expanding by minors of those elements, it will be readily seen that some of the terms disappear because of multiplication by zero. It is possible to make some zeros appear in a column or row by applying a well-established property of determinants.

PROPERTY. *The value of a determinant will remain unchanged if the elements of any column or row are multiplied by the same quantity, k, and then added to the corresponding elements of any other column or row.*

Let us apply this to the solution of Example 4–23 in which there are some zero coefficients:

$$\begin{aligned} 10.5I_1 + \ \ 0I_2 + 2I_3 &= 12 \\ 0I_1 + 8.4I_2 + 2I_3 &= 10 \\ I_1 + \ \ \ I_2 - \ \ I_3 &= 0 \end{aligned}$$

The third order determinant here is:

$$\begin{vmatrix} 10.5 & 0 & 2 \\ 0 & 8.4 & 2 \\ 1 & 1 & -1 \end{vmatrix}$$

From this:

$$I_1 = \frac{\begin{vmatrix} 12 & 0 & 2 \\ 10 & 8.4 & 2 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 10.5 & 0 & 2 \\ 0 & 8.4 & 2 \\ 1 & 1 & -1 \end{vmatrix}}$$

By multiplying the elements of the third row in the numerator by -8.4, we get 0 -8.4 $+8.4$ and adding these to the second row elements gives 10 0 10.4. Then our numerator becomes:

$$\begin{vmatrix} 12 & 0 & 2 \\ 10 & 0 & 10.4 \\ 0 & 1 & -1 \end{vmatrix}$$

Again by multiplying the elements of the third row in the denominator by -8.4 and adding to the second row elements our denominator becomes:

$$\begin{vmatrix} 10.5 & 0 & 2 \\ -8.4 & 0 & 10.4 \\ 1 & 1 & -1 \end{vmatrix}$$

Expanding by minors of the second column elements:

$$I_1 = \frac{\begin{vmatrix} 12 & 0 & 2 \\ 10 & 0 & 10.4 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 10.5 & 0 & 2 \\ -8.4 & 0 & 10.4 \\ 1 & 1 & 1 \end{vmatrix}}$$

$$= \frac{-(0)\begin{vmatrix} 10 & 10.4 \\ 0 & -1 \end{vmatrix} + (0)\begin{vmatrix} 12 & 2 \\ 0 & -1 \end{vmatrix} - (1)\begin{vmatrix} 12 & 2 \\ 10 & 10.4 \end{vmatrix}}{-(0)\begin{vmatrix} -8.4 & 10.4 \\ 1 & -1 \end{vmatrix} + (0)\begin{vmatrix} 10.5 & 2 \\ 1 & -1 \end{vmatrix} - (1)\begin{vmatrix} 10.5 & 2 \\ -8.4 & 10.4 \end{vmatrix}}$$

Since the first two terms in both the numerator and denominator are each multiplied by zero, they may be dropped entirely. Therefore:

$$I_1 = \frac{-(1)\begin{vmatrix} 12 & 2 \\ 10 & 10.4 \end{vmatrix}}{-(1)\begin{vmatrix} 10.5 & 2 \\ -8.4 & 10.4 \end{vmatrix}}$$

$$= \frac{(-1)(124.8 - 20)}{(-1)(109.2 + 16.8)} = \frac{(-1)(104.8)}{(-1)(126)} = \frac{-104.8}{-126} = 0.832 \text{ amp.}$$

In solving for I_2 the denominator will be same as for I_1 and will be denoted by D, since it need not be transformed again. The numerator is transformed by multiplying the third row by -10.5 and adding to the first row. Then the expansion of the numerator is made by minors of the first column elements.

$$I_2 = \frac{\begin{vmatrix} 10.5 & 12 & 2 \\ 0 & 10 & 2 \\ 1 & 0 & -1 \end{vmatrix}}{\text{D}} = \frac{\begin{vmatrix} 0 & 12 & 12.5 \\ 0 & 10 & 2 \\ 1 & 0 & -1 \end{vmatrix}}{-126} = \frac{(1)\begin{vmatrix} 12 & 12.5 \\ 10 & 2 \end{vmatrix}}{-126}$$

$$= \frac{(1)(24 - 125)}{-126} = \frac{-101}{-126} = +0.802 \text{ amp.}$$

In a similar fashion, to solve for I_3, multiply the third row by -8.4 and add to the second row. Expand by minors of the second column.

$$I_3 = \frac{\begin{vmatrix} 10.5 & 0 & 12 \\ 0 & 8.4 & 10 \\ 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 10.5 & 0 & 2 \\ 0 & 8.4 & 2 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{\begin{vmatrix} 10.5 & 0 & 12 \\ -8.4 & 0 & 10 \\ 1 & 1 & 0 \end{vmatrix}}{-126} = \frac{-(1)\begin{vmatrix} 10.5 & 12 \\ -8.4 & 10 \end{vmatrix}}{-126}.$$

$$= \frac{(-1)(105 + 100.8)}{-126} = \frac{-205.8}{-126} = 1.634 \text{ amp.}$$

We have made use of one of the properties of determinants to simplify the solution of three simultaneous equations. However, there are several properties that are important and we shall enumerate them including the one already used.

PROPERTY 1. *The value of the determinant is zero if each element of any row or column is zero.* Thus $\begin{vmatrix} 4 & 0 & -1 \\ 2 & 0 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 0$ because the elements of the second column are 0.

PROPERTY 2. *In any determinant, when corresponding rows and columns are interchanged the value of the determinant remains unchanged.* Thus

$$\begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} = \begin{vmatrix} a_1a_2a_3 \\ b_1b_2b_3 \\ c_1c_2c_3 \end{vmatrix}.$$

Property 3. *A factor common to all elements of any row or column is a common factor of the determinant.* Thus

$$\begin{vmatrix} a_1kb_1c_1 \\ a_2kb_2c_2 \\ a_3kb_3c_3 \end{vmatrix} = k \begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix}.$$

Property 4. *The sign of the determinant is changed if any two rows or columns of the determinant are interchanged.* Thus

$$\begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} = - \begin{vmatrix} b_1a_1c_1 \\ b_2a_2c_2 \\ b_3a_3c_3 \end{vmatrix}.$$

Property 5. *The value of a determinant will remain unchanged if the elements of any column or row are multiplied by the same quantity, k, and then added to the corresponding elements of any other column or row.* Thus

$$\begin{vmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \end{vmatrix} = \begin{matrix} a_1 & b_1 & c_1 \\ a_2 + ka_3, & b_2 + kb_3, & c_2 + kc_3. \\ a_3 & b_3 & c_3 \end{matrix}$$

Property 6. *The value of a determinant is zero if any two rows or columns of the determinant are identical.*

$$\begin{vmatrix} a_1b_1a_1 \\ a_2b_2a_2 \\ a_3b_3a_3 \end{vmatrix} = 0.$$

19. Solution of determinants of the fourth and higher orders. The solution of four simultaneous equations by determinants can be carried out in the same manner by minors as shown for three equations. However, as the number of equations increases, the number of elements in the determinant also increases and the process of evaluation becomes longer. The following example is typical of a fourth order determinant solution.

Example 4–25. Solve the following simultaneous equations by minors:

$$\begin{aligned} a + b - c + d &= 7, \\ 2a - b + 3c + 2d &= 3, \\ 3a + 2b - 2c - 4d &= -3, \\ -5a + 3b - 4c - d &= 2. \end{aligned}$$

Solution:

$$a = \frac{\begin{vmatrix} +7 & +1 & -1 & +1 \\ +3 & -1 & +3 & +2 \\ -3 & +2 & -2 & -4 \\ +2 & +3 & -4 & -1 \end{vmatrix}}{\begin{vmatrix} +1 & +1 & -1 & +1 \\ +2 & -1 & +3 & +2 \\ +3 & +2 & -2 & -4 \\ -5 & +3 & -4 & -1 \end{vmatrix}}.$$

Expanding by minors of the first column the numerator becomes:

$$7\begin{vmatrix} -1 & 3 & 2 \\ 2 & -2 & -4 \\ 3 & -4 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 & 1 \\ 2 & -2 & -4 \\ 3 & -4 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & -1 & 1 \\ -1 & 3 & 2 \\ 3 & -4 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 & 1 \\ -1 & 3 & 2 \\ 2 & -2 & -4 \end{vmatrix}.$$

Further expansion of the third order determinants by minors of the first columns gives:

$$\begin{aligned}
&7\left[-1\begin{vmatrix} -2 & -4 \\ -4 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ -4 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ -2 & -4 \end{vmatrix}\right] \\
&\quad - 3\left[1\begin{vmatrix} -2 & -4 \\ -4 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ -4 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ -2 & -4 \end{vmatrix}\right] \\
&\quad - 3\left[1\begin{vmatrix} 3 & 2 \\ -4 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ -4 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix}\right] \\
&\quad - 2\left[1\begin{vmatrix} 3 & 2 \\ -2 & -4 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ -2 & -4 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix}\right] \\
&\quad = -7(+2 - 16) - 14(-3 + 8) + 21(-12 + 4) \\
&\quad - 3(2 - 16) + 6(1 + 4) - 9(4 + 2) - 3(-3 + 8) \\
&\quad - 3(1 + 4) - 9(-2 - 3) - 2(-12 + 4) - 2(4 + 2) \\
&\quad -4(-2 - 3) = +98 - 70 - 168 + 42 + 30 \\
&\qquad - 54 - 15 - 15 + 45 + 16 - 12 + 20 = -83.
\end{aligned}$$

The denominator expanded gives:

$$1\begin{vmatrix} -1 & 3 & 2 \\ 2 & -2 & -4 \\ 3 & -4 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 & 1 \\ 2 & -2 & -4 \\ 3 & -4 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 & 1 \\ -1 & 3 & 2 \\ 3 & -4 & -1 \end{vmatrix} - (-5)\begin{vmatrix} 1 & -1 & 1 \\ -1 & 3 & 2 \\ 2 & -2 & -4 \end{vmatrix}.$$

Since we have expanded by minors of the first column, the third order determinates are the same as in the numerator and their expansion will also be the same. So we have:

$$\begin{aligned}
&1[-1(-14) - 2(+5) + 3(-8)] - 2[1(-14) - 2(+5) + 3(+6)] \\
&+ 3[1(+5) + 1(+5) + 3(-5)] + 5[1(-8) + 1(+6) + 2(-5)] \\
&= 14 - 10 - 24 + 28 + 20 - 36 + 15 + 15 \\
&\qquad - 45 - 40 + 30 - 50 = -83.
\end{aligned}$$

Therefore:
$$a = \frac{-83}{-83} = +1$$

$$b = \frac{\begin{vmatrix} 1 & 7 & -1 & 1 \\ 2 & 3 & 3 & 2 \\ 3 & -3 & -2 & -4 \\ -5 & 2 & -4 & -1 \end{vmatrix}}{D}$$

$$= \frac{1\begin{vmatrix} 3 & 3 & 2 \\ -3 & -2 & -4 \\ 2 & -4 & -1 \end{vmatrix} - 2\begin{vmatrix} 7 & -1 & 1 \\ -3 & -2 & -4 \\ 2 & -4 & -1 \end{vmatrix} + 3\begin{vmatrix} 7 & -1 & 1 \\ 3 & 3 & 2 \\ 2 & -4 & -1 \end{vmatrix} - (-5)\begin{vmatrix} 7 & -1 & 1 \\ 3 & 3 & 2 \\ -3 & -2 & -4 \end{vmatrix}}{D}$$

The numerator is:

$$1\left[3\begin{vmatrix} -2 & -4 \\ -4 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ -4 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ -2 & -4 \end{vmatrix}\right]$$
$$- 2\left[7\begin{vmatrix} -2 & -4 \\ -4 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ -4 & -1 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ -2 & -4 \end{vmatrix}\right]$$
$$+ 3\left[7\begin{vmatrix} 3 & 2 \\ -4 & -1 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 \\ -4 & -1 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix}\right]$$
$$+ 5\left[7\begin{vmatrix} 3 & 2 \\ -2 & -4 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 \\ -2 & -4 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix}\right]$$
$$= 3(2 - 16) + 3(-3 + 8) + 2(-12 + 4) - 14(2 - 16)$$
$$- 6(1 + 4) - 4(4 + 2) + 21(-3 + 8) - 9(1 + 4)$$
$$+ 6(-2 - 3) + 35(-12 + 4) - 15(4 + 2)$$
$$- 15(-2 - 3) = -42 + 15 - 16 + 196 - 30 - 24$$
$$+ 105 - 45 - 30 - 280 - 90 + 75 = -166$$

Then:
$$b = \frac{-166}{D} = \frac{-166}{-83} = 2.$$

The values for c and d can be found in a similar manner. It is left to the student to complete this solution.

Example 4–26. An electrical network of resistances supplied by a battery gives the following simultaneous equations to solve:

$$\begin{aligned} 0I + I_1 + I_2 - I_T &= 0, \\ I - I_1 + I_2 + 0I_T &= 0, \\ I + 4I_1 - 2I_2 + 0I_T &= 0, \\ 0I + 4I_1 + 2I_2 + 0I_T &= 8.25. \end{aligned}$$

Solution: Expanding by minors of the fourth column elements:

First Form | *Second Form* | *Final Form*

$$D = \begin{vmatrix} 0 & +1 & +1 & -1 \\ 1 & -1 & +1 & 0 \\ 1 & +4 & -2 & 0 \\ 0 & +4 & +2 & 0 \end{vmatrix} = -(-1)\begin{vmatrix} 1 & -1 & +1 \\ 1 & +4 & -2 \\ 0 & +4 & +2 \end{vmatrix} = +1\begin{vmatrix} 1 & -1 & +1 \\ 0 & +5 & -3 \\ 0 & +4 & +2 \end{vmatrix}.$$

The final form is obtained by multiplying the first row of the second form by (-1) and adding the result to the second row of the second form.

Then expanding this final form by first column elements:

$$D = (+1)(+1)\begin{vmatrix} 5 & -3 \\ 4 & +2 \end{vmatrix} = 10 - (-12) = 22.$$

$$\begin{array}{cccccc} & \textit{First Form} & & \textit{Second Form} & & \textit{Final Form} \\ I_T = & \dfrac{\begin{vmatrix} 0 & +1 & +1 & -0 \\ 1 & -1 & +1 & 0 \\ 1 & +4 & -2 & 0 \\ 0 & 4 & +2 & 8.25 \end{vmatrix}}{D} & = & \dfrac{+8.25\begin{vmatrix} 0 & +1 & +1 \\ 1 & -1 & +1 \\ 1 & +4 & -2 \end{vmatrix}}{D} & = & \dfrac{8.25\begin{vmatrix} 0 & +1 & +1 \\ 1 & -1 & +1 \\ 0 & +5 & -3 \end{vmatrix}}{D}. \end{array}$$

The final form is obtained by multiplying the second row of the second form by (-1) and adding the result to the third row of the second form.

Then expanding this final form by first column elements:

$$I_T = \frac{8.25(-1)\begin{vmatrix} 1 & 1 \\ 5 & -3 \end{vmatrix}}{D} = \frac{-8.25(-3-5)}{22} = \frac{66}{22} = 3 \text{ amps.}$$

In like manner:

$$I = \frac{\begin{vmatrix} 0 & 1 & 1 & -1 \\ 0 & -1 & +1 & 0 \\ 0 & +4 & -2 & 0 \\ 8.25 & +4 & +2 & 0 \end{vmatrix}}{D} = \frac{-8.25\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 4 & -2 & 0 \end{vmatrix}}{D}.$$

$$= \frac{(-8.25)(-1)\begin{vmatrix} -1 & +1 \\ +4 & -2 \end{vmatrix}}{D}$$

$$= \frac{8.25(2-4)}{22} = -\frac{16.5}{22} = -0.75 \text{ amps.}$$

$$I_1 = \frac{\begin{vmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 8.25 & 2 & 0 \end{vmatrix}}{D} = \frac{-(-1)\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 8.25 & 2 \end{vmatrix}}{D}.$$

$$= \frac{1(-8.25)\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}}{D} = \frac{24.75}{22} = 1.125 \text{ amps.}$$

$$I_2 = \frac{\begin{vmatrix} 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 4 & 8.25 & 0 \end{vmatrix}}{D} = \frac{-(-1)\begin{vmatrix} 1 & -1 & 0 \\ 1 & 4 & 0 \\ 0 & 4 & 8.25 \end{vmatrix}}{D}.$$

$$= \frac{(+1)(+8.25)\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix}}{D} = \frac{41.25}{22} = 1.875 \text{ amps.}$$

As a check, substituting back in two of the original equations:

$$0 + I_1 + I_2 - I_T = 0,$$
$$0 + 1.125 + 1.875 - 3 = 0 \text{ (check)};$$

and

$$I - I_1 + I_2 + 0 = 0,$$
$$-0.75 - 1.125 + 1.875 + 0 = 0 \text{ (check)}.$$

EXERCISE 4-4

Solve the following problems by determinants:

1. $2m + n = 7,$
   $m + 2n = -4.$
2. $3x - y = 6,$
   $x + 2y = 16.$
3. $4x - 3y + z = 3,$
   $3x + 2y - 3z = 0,$
   $x + y - z = 1.$
4. $x + y + z = 4,$
   $2x - y - 3z = -5,$
   $3x + 2y + 4z = 21.$
5. $4I_1 + 3I_3 = 10.5,$
   $5I_2 + 3I_3 = 24.$
   $I_1 + I_2 - I_3 = 0.$

20. Quadratic equations. Thus far we have considered only simple equations or equations of the first order or degree. The simple equation $x - 5 = 0$ is one of the first order and has one root, $x = 5$. Similarly, the equation $x - 2 = 0$ is a first-order equation and has the one root $x = 2$. The product of these two simple equations, $x^2 - 7x + 10 = 0$, is an equation of the second order, since it contains the square of the unknown as its highest power. Such an equation is called a *quadratic equation.*

A quadratic equation may be one of two types, a *pure quadratic* or an *affected quadratic.* A pure quadratic is one that contains only the second power of the unknown and an affected quadratic is one that contains both the second power and the first power of the unknown. Thus $4x^2 = 16$ is a pure quadratic and $2x^2 + 13x + 15 = 0$ is an affected quadratic.

Any pure quadratic equation has two roots, opposite in sign and numerically equal. In order to solve a pure quadratic, the term containing the square of the unknown is made the left member of the equation and all other terms the right member of the equation. Then the square root of both sides is extracted.

EXAMPLE 4-27. Solve the equation $2x^2 - 5 = 67$.

Solution:

$$2x^2 - 5 = 67,$$
$$2x^2 = 67 + 5 = 72,$$
$$x^2 = 36,$$
$$x = \pm 6.$$

(*Note:* The $\pm$ sign, read "plus or minus," indicates that the answer is either a plus 6 or a minus 6. This is true because either $+6$ or -6 may be squared to give a $+36$. Both answers should be given unless it is evident from the statement of the problem that one of them is impossible. This can be ascertained by substituting each value in the original equation.)

An affected quadratic equation can always be reduced to the type form $ax^2 + bx + c = 0$ and may be solved by factoring or by completing the square.

21. Solution of a quadratic by factoring. If the trinomial equation $ax^2 + bx + c = 0$ can be factored into two linear factors, each factor may be set equal to zero and solved for the unknown, because the product of the two factors is zero.

EXAMPLE 4–28. Solve the equation $x^2 - 8x + 15 = 0$.

Solution: By factoring $x^2 - 8x + 15 = (x - 5)(x - 3)$.

Thus, since $x^2 - 8x + 15 = 0$,

then $(x - 5)(x - 3) = 0$.

Therefore, $x - 5 = 0$

or $x = 5$

and $x - 3 = 0$

or $x = 3$.

22. Solution of a quadratic by completing the square. If the trinomial equation $ax^2 + bx + c = 0$ cannot be factored easily, a general method known as the *method of completing the square* may be used. In this method, all terms containing the unknown are placed in the left member of the equation and all constant terms in the right member of the equation. Then the equation is arranged so that the left member becomes a perfect square, and the square root of each member is finally taken.

EXAMPLE 4–29. Solve the equation $x^2 - 5x - 14 = 0$ by completing the square.

Solution: Rearrange the terms, $x^2 - 5x = 14$. Now, to make the left member a perfect square, some value must be added to it. This value is found by taking half the coefficient of x and squaring it. Then it is added to both sides of the equation and a square root taken.

$$x^2 - 5x + \left(\frac{5}{2}\right)^2 = 14 + \left(\frac{5}{2}\right)^2,$$

$$x - \frac{5}{2} = \pm\sqrt{14 + \left(\frac{5}{2}\right)^2},$$

$$x = \frac{5}{2} \pm \sqrt{14 + \frac{25}{4}},$$

$$x = \frac{5}{2} \pm \sqrt{\frac{81}{4}},$$

$$x = \frac{5}{2} \pm \frac{9}{2},$$

$$x = \frac{14}{2} \text{ or } \frac{-4}{2},$$

$$x = 7 \text{ or } -2.$$

If the coefficient of the squared term is not unity, it must be made unity before completing the square. Therefore, the complete equation must be divided by the coefficient of the squared term.

EXAMPLE 4–30. Solve the equation $2x^2 - 13x + 15 = 0$.

Solution: Transposing, $2x^2 - 13x = -15;$

dividing by the coefficient of x^2, $x^2 - \frac{13}{2}x = -\frac{15}{2};$

adding the square of half the coefficient of x to both sides, $x^2 - \frac{13}{2}x + \left(\frac{13}{4}\right)^2 = -\frac{15}{2} + \left(\frac{13}{4}\right)^2;$

taking square roots, $x - \frac{13}{4} = \pm\sqrt{-\frac{15}{2} + \frac{169}{16}};$

solving for x, $x = \frac{13}{4} \pm \sqrt{\frac{-120 + 169}{16}},$

or $x = \frac{13}{4} \pm \frac{7}{4} = \frac{20}{4}, \text{ or } \frac{6}{4}.$

Therefore, $x = 5, \text{ or } 1\frac{1}{2}.$

We can now formulate rules for solving equations by completing the squares:

1. Put the equation in the form $x^2 + bx = c$ and make the coefficient of x^2 unity if it is not already unity, by dividing the whole equation by that coefficient.

2. Add to both sides of the equation the square of half the coefficient of x. This makes the left member of the equation a perfect square.

3. Extract the square root of each member of the equation and remember to place a $\pm$ sign in front of the square root of the right member. The square root of the left member will be the respective square roots of the first and third terms connected by the sign of the middle term.

4. Solve the two equations thus formed for the unknown.

5. Check the results by substitution of the values in the original equation.

23. The quadratic formula. A method of solution for all affected quadratic equations can be developed by applying the method of completing the square, as given in Article 22, to the general quadratic equation $ax^2 + bx + c = 0$. The quadratic formula is obtained from this procedure.

EXAMPLE 4–31. Solve $ax^2 + bx + c = 0$ for x.

Solution: Transposing, $ax^2 + bx = -c;$

dividing by a, the coefficient of x^2, $x^2 + \frac{b}{a}x = -\frac{c}{a};$

adding the square of half the coefficient of x to both sides, $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a};$

taking square roots, $x + \dfrac{b}{2a} = \pm \sqrt{\left(\dfrac{b}{2a}\right)^2 - \dfrac{c}{a}}$;

simplifying, $$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

Then, $$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

or $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula gives us an expression that can be used for the solution of any quadratic equation, because the value of x is determined from the values of the coefficients and the constant term. It should be noted that a is the coefficient of the squared term, b is the coefficient of the first power term, and c is the known constant term. Let us apply it to a problem.

EXAMPLE 4–32. Solve the equation $2x^2 - 13x + 15 = 0$ by the quadratic formula.

Solution: $a = 2$, $b = -13$, and $c = 15$. Therefore,

$$x = \frac{-(-13) \pm \sqrt{(-13)^2 - (4)(2)(15)}}{(2)(2)},$$

$$x = \frac{13 \pm \sqrt{169 - 120}}{4} = \frac{13 \pm \sqrt{49}}{4} = \frac{13 \pm 7}{4},$$

$$x = \frac{20}{4}, \text{ or } \frac{6}{4},$$

or $x = 5$, or $1\frac{1}{2}$ (which checks the answer to Example 4–30).

There are times when an equation may not appear at the first glance to be a quadratic equation. Thus, $x^4 - 5x^2 + 4 = 0$ may not look like a quadratic equation, but it is one because the fourth-power term is the square of the second-power term and no third-power or first-power term is present. Therefore, we may say that any equation is a quadratic that contains, besides a constant term, only two powers of the unknown, with one power the square of the other. Since the equation $x^4 - 5x^2 + 4 = 0$ fulfills these conditions, it is a quadratic. It is first solved for x^2 and then a square root taken to get x.

EXAMPLE 4–33. Solve $x^4 - 5x^2 + 4 = 0$ for x.

Solution: Solving for x^2 by the formula,

$$x^2 = \frac{-(-5) \pm \sqrt{(-5)^2 - (4)(1)(4)}}{(2)(1)}$$

$$x^2 = \frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm 3}{2} = 4 \text{ or } 1,$$

$$x = \pm 2 \text{ or } \pm 1.$$

24. Nature of the roots of a quadratic. We have found that the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

When a, b, and c are real numbers, the expression $\sqrt{b^2 - 4ac}$ may have any value, positive, negative, or zero. Therefore, it is called the *discriminant of the equation.* If $\sqrt{b^2 - 4ac}$ is positive, the two roots are real and unequal; if $\sqrt{b^2 - 4ac}$ is negative, the two roots are imaginary, because the square root of a negative quantity cannot be taken; and if $\sqrt{b^2 - 4ac}$ is zero, the two roots are equal and real. If $\sqrt{b^2 - 4ac}$ is a perfect square, the two roots are rational; but if it is not a perfect square, the two roots are irrational.

Checking the roots of a quadratic equation by substituting them in the original expression is often inconvenient and it is helpful to know what the *sum* or the *product* of the roots will be in terms of a, b, and c.

Thus: $$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = \frac{-b}{a}.$$

Also: $$x_1x_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$
$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{c}{a}.$$

Thus, before solving any quadratic, the sum of the roots and the product of the roots is known. In Example 4–32, the sum of the roots of the equation $2x^2 - 13x + 15 = 0$ is:

$$x_1 + x_2 = \frac{-b}{a} = \frac{-(-13)}{2} = \frac{13}{2} = 6\tfrac{1}{2},$$

and their product is:

$$x_1x_2 = \frac{c}{a} = \frac{15}{2} = 7\tfrac{1}{2}.$$

Since we found $x = 5$ or $1\tfrac{1}{2}$, we know the roots are correct because the sum and product prove them true.

25. Simultaneous equations of second order. The solution of simultaneous equations when one of them is a second order may present some difficulties. In general, the method of attack will be indicated by a careful study of the equations. Quite frequently the method of substitution will serve more satisfactorily than any of the other methods. A complete discussion of the solution of such equations will not be attempted here since the solution often involves third powers or higher and such solutions are beyond the scope of this work.

The following examples are presented to illustrate some of the more simple cases.

EXAMPLE 4–34. Solve the simultaneous equations

$$x^2 + y^2 = 50,$$
$$xy = 7.$$

Solution: Solving the second equation for y in terms of x and substituting in the first gives

$$y = \frac{7}{x},$$

$$x^2 + \left(\frac{7}{x}\right)^2 = 50.$$

Then
$$x^2 + \frac{49}{x^2} = 50,$$
$$x^4 + 49 = 50x^2,$$
or
$$x^4 - 50x^2 + 49 = 0.$$

This can be solved for x^2 by factoring:

$$(x^2 - 49)(x^2 - 1) = 0.$$

Therefore,
$$x^2 - 49 = 0,$$
$$x^2 = 49,$$
$$x = \pm 7.$$
Also
$$x^2 - 1 = 0,$$
$$x^2 = 1,$$
$$x = \pm 1.$$

If $x = \pm 7$, by substitution in the second equation $y = \pm 1$. And if $x = \pm 1$, by substitution in the second equation $y = \pm 7$. Results can be checked by substitution in the first equation.

EXAMPLE 4–35. Solve the simultaneous equations

$$x^2 + y^2 = 8,$$
$$x^2 - xy + y^2 = 4.$$

Solution: Subtracting the second from the first,

$$\begin{array}{rcrcrcr} x^2 & & & + & y^2 & = & 8 \\ +x^2 & - & xy & + & y^2 & = & +4 \\ - & + & & - & & & - \\ \hline & & xy & & & = & 4 \end{array}$$

or
$$y = \frac{4}{x}.$$

By substitution in the first equation,

$$x^2 + \left(\frac{4}{x}\right)^2 = 8.$$

Then
$$x^2 + \frac{16}{x^2} = 8,$$
$$x^4 + 16 = 8x^2,$$
$$x^4 - 8x^2 + 16 = 0.$$
Factoring,
$$(x^2 - 4)(x^2 - 4) = 0.$$
Therefore,
$$x^2 - 4 = 0,$$
$$x^2 = 4,$$
and
$$x = \pm 2.$$
By substitution in the first equation, $y = \pm 2$.

26. Solution of higher order equations by synthetic substitution. The processes of synthetic division can be extended to determine the real roots or to divide out the factors of an equation and is especially adaptable where the given equation is the third degree or higher.

EXAMPLE 4–36. Find the roots of the equation $y = x^4 - 4x^3 - 7x^2 + 22x + 24$.

Solution: Choose a factor of the constant term such that a value of zero will be obtained for the equation when the factor is substituted for x. Thus, when -1 is substituted for x there results $1 + 4 - 7 - 22 + 24$, which is equal to zero. Therefore, $x - (-1)$ or $x + 1$ is a factor of the original equation. The division of the original by this factor can be performed by synthetic division and the same procedure followed to get all the other factors of which there may be as many as four since the original equation is the fourth power.

Thus:

$$\begin{array}{lllll|l}
1 - 4 & - \;\; 7 & + 22 & + 24 & & \underline{+1} \\
\;\;\; +1 & - \;\; 5 & - \;\; 2 & + 24 & & \\
\;\;\; - \;\; + & \;\;\; + & \;\;\; - & & & \\
\hline
1 - 5 & - \;\; 2 & + 24 & + \;\; 0 & & \underline{2} \\
\;\;\; +2 & - 14 & + 24 & & & \\
\;\;\; - \;\; + & \;\;\; - & & & & \\
\hline
1 - 7 & + 12 & + \;\; 0 & & & \underline{-3} \\
\;\;\; -3 & + 12 & & & & \\
\;\;\; + \;\; - & & & & & \\
\hline
1 - 4 & + \;\; 0 & & & & \underline{-4} \\
\;\;\; -4 & & & & & \\
\;\;\; + & & & & & \\
\hline
1 + 0 & & & & &
\end{array}$$

Hence the four factors become $x + 1$, $x + 2$, $x - 3$, and $x - 4$. The product of these four factors will result in the original equation, or we may write $y = (x + 1)(x + 2)(x - 3)(x - 4)$ in place of the original. Since these factors were found by substituting values for x that made the

original equation equal to zero, then we may also write

$$(x + 1)(x + 2)(x - 3)(x - 4) = 0.$$

Now each factor may be set equal to zero and solved for x, giving the four roots, -1, -2, $+3$, and $+4$, values for x.

It will be noted that the solution uses synthetic substitution for finding each root. The first trial divisor, $+1$, means that the original expression, $x^4 - 4x^3 - 7x^2 + 22x + 24$, is being divided by $x + 1$. The result of this division is $1 - 5 - 2 + 24 + 0$ which means $x^3 - 5x^2 - 2x + 24$ with a remainder of zero. Since the division gives an even result with a remainder of zero, -1 becomes a real root. If the remainder had been other than zero, the trial divisor would not be a real root. The second trial divisor, $+2$, means that the result of the first division, $x^3 - 5x^2 - 2x + 24$, is being divided by $x + 2$. This gives $x^2 - 7x + 12$ with a remainder of zero and, therefore, -2 is a second real root. The third trial divisor, -3, means that the result of the second division, $x^2 - 7x + 12$ is being divided by $x - 3$. This gives $x - 4$ with a remainder of zero and, therefore, $+3$ is a third real root. Finally, the fourth trial divisor, -4, means that $x - 4$ is being divided by $x - 4$ which gives $+1$ with a remainder of zero and $+4$ becomes a fourth real root. It should be noted that the *sign of the root* is opposite the *sign of the divisor*. Thus, the divisor $x + 1$ when solved for x gives $x = -1$ as a root.

EXAMPLE 4–37. Find the roots of $y = x^3 - 13x + 12$ by synthetic substitution.

Solution: Substitution of $+1$, $+3$, or -4 will make the expression equal to zero. Therefore, by synthetic substitution we have:

$$\begin{array}{rrrrl}
1 & +0 & -13 & +12 & \lfloor\underline{-1} \\
 & -1 & -1 & +12 & \\
 & + & + & - & \\
\hline
1 & +1 & -12 & +0 & \lfloor\underline{-3} \\
 & -3 & -12 & & \\
 & + & + & & \\
\hline
1 & +4 & +0 & & \lfloor\underline{+4} \\
 & +4 & & & \\
 & - & & & \\
\hline
1 & +0 & & &
\end{array}$$

Since the remainder for each division is zero the roots are $+1$, $+3$, and -4. The factors of the original expression are $(x - 1)$, $(x - 3)$, and $(x + 4)$. *Note that a zero has been used* for a coefficient of the missing term in x^2.

EXAMPLE 4–38. Find the whole number roots (if any) of $y = x^3 - 4x^2 + 6x - 5$.

Solution (by synthetic substitution):

$$\begin{array}{rrrr|l} 1 & -4 & +6 & -5 & -1 \\ & -1 & +3 & -3 & \\ & + & - & + & \\ \hline 1 & -3 & +3 & -2 & \end{array} \qquad \begin{array}{rrrr|l} 1 & -4 & +6 & -5 & +1 \\ & +1 & -5 & +11 & \\ & - & + & - & \\ \hline 1 & -5 & +11 & -16 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & -4 & +6 & -5 & -5 \\ & -5 & -5 & -55 & \\ & + & + & + & \\ \hline 1 & +1 & +11 & +50 & \end{array} \qquad \begin{array}{rrrr|l} 1 & -4 & +6 & -5 & +5 \\ & +5 & -45 & +255 & \\ & - & + & - & \\ \hline 1 & -9 & +51 & -260 & \end{array}$$

By using the several factors of 5 as trial divisors, we find that, in each trial, there is a remainder that becomes larger numerically as the trial divisor is made larger. Therefore, it is evident that there are no whole number roots of this expression.

EXERCISE 4–5

Solve the following pure quadratic equations:

1. $4x^2 + 5 = 69$.
2. $5x^2 + 12 = 137$.
3. $3x^2 = 27$.
4. $15x^2 - 14 = 1$.
5. $6x^2 + 21 = 885$.
6. $-6x^2 + 740 = 14$.
7. $8 = 53 - 5x^2$.
8. $10x^2 + 60 = 1{,}500$.
9. $46 = 14x^2 - 10$.
10. $117 = 17 + 4x^2$.

Solve the following affected quadratics by factoring:

11. $x^2 + 2x - 15 = 0$.
12. $x^2 - 12x + 27 = 0$.
13. $x^2 + x - 30 = 0$.
14. $6x^2 + 72x + 216 = 0$.
15. $84x^2 - 37x - 30 = 0$.
16. $x^4 - 34x^2 + 225 = 0$.
17. $6x^2 = 48x - 96$.
18. $x^2 + 3a^2 = 4ax$.
19. $65x^2 - 784 = x^4$.
20. $x^2 + ax = bx + ab$.

Solve the following affected quadratics by completing the square:

21. $x^2 - 6x - 16 = 0$.
22. $x^2 - 160 - 6x = 0$.
23. $x^2 + 105 = 26x$.
24. $6 - 5x = 6x^2$.
25. $5x^2 = 18x + 8$.
26. $0.2x^2 = 3.5 - 0.3x$.
27. $\dfrac{1}{x+1} - \dfrac{10}{3} + \dfrac{3}{x-1} = 0$.
28. $\dfrac{1}{a-x} - \dfrac{1}{a+x} - \dfrac{3-x^2}{a^2-x^2} = 0$.
29. $-10 = 9x - 7x^2$.
30. $x^2 - 4ab + 2ax - 2bx = 0$.

Solve the following by use of the quadratic formula:

31. $x^2 - 10x + 24 = 0$.
32. $y^2 - y - 20 = 0$.
33. $2A^2 + 352 = 54A$.
34. $x^2 - 7x = 494$.
35. $E^2 = \left(1\frac{1}{4} - E\right)^2 - \left(E + \frac{3}{4}\right)^2$.
36. $x^2 - ab - ax + bx = 0$.

37. A merchant received \$80 for a number of yards of cloth. If the number of dollars a yard is equal to $5/16$ of the number of yards, how many yards were sold?

38. In an alternating current parallel circuit the following expression is found:

$$g = \frac{R}{R^2 + x^2}.$$

Solve the equation for R and determine its value when $g = 0.04$ and $x = 12$.

39. It takes 4 hr 12 min for the crew of a boat to row 12 miles down a river with the current and back again against the current. At what rate can the crew row in still water, if the speed of the current is 3 mph?

40. The distance passed over by a freely falling body with an initial velocity of V_0 is shown by the equation $d = V_0t + \frac{1}{2}gt^2$, where d is the distance in feet, V_0 is the velocity in feet per second, t is the time in seconds, and g is the acceleration due to gravity in feet per second per second. Solve the equation for t and find the time it will take a body to reach the ground from a height of 656 ft if it is given an initial velocity of 100 ft per second. Use $g = 32$.

41. The center of gravity for a cylindrical surface with one end closed is given by the equation

$$a = \frac{2h^2}{4h + d}.$$

Solve the equation for h and determine its value if $a = 6$ in. and $d = 10$ in.

42. The loss of head of water flowing in a pipe is given by the equation

$$H = \left(\frac{L}{d}\right)\left(\frac{4V^2 + 5V - 2}{1200}\right).$$

Solve the equation for the velocity, V.

43. Water enters a 40-gal tank through one pipe and is discharged through another. In 4 min, 1 gal more is discharged than is brought in. The filler pipe can fill the tank in 3 min less time than it takes the discharge pipe to discharge 66 gal. How long does it take to fill the tank?

44. If a body is thrown vertically upward with an initial velocity of V_0, the distance d from its starting point after any time t is given by the expression

$$d = V_0t - \frac{1}{2}gt^2, \text{ where } g = 32 \text{ (approx.)}.$$

If the body is thrown vertically downward, the above equation becomes

$$d = V_0t + \frac{1}{2}gt^2.$$

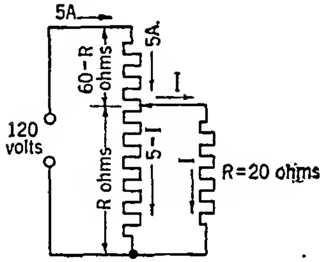
If no initial velocity is imparted to the body, V_0 becomes zero and the equations become

$$d = -\frac{1}{2}gt^2 \text{ (upward)},$$
$$d = +\frac{1}{2}gt^2 \text{ (downward)}.$$

Also, the distance covered by sound in air is represented by the equation $d = Vt$ where V is the velocity in feet per second and t is the time in seconds.

Determine the depth of a well if the sound of a stone striking the bottom is heard 5 sec after the stone is dropped. The velocity of sound in air is approximately 1,100 ft per second.

(*Note:* 5 sec is the time that it takes the stone to drop plus the time it takes for the sound to return.)



45. The diagram at the left is the electrical connection for a potentiometer circuit. The equations according to Kirchhoff's laws will be

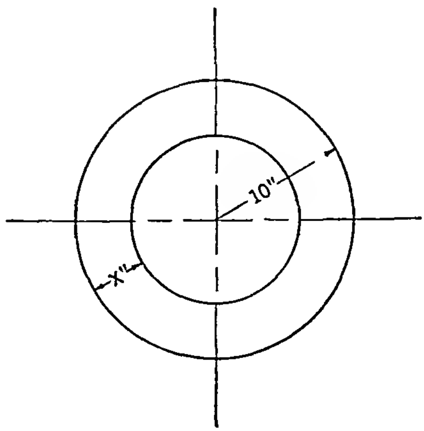
$$120 - 5(60 - R) - (5 - I)(R) = 0.$$
$$120 - 5(60 - R) - 20I = 0.$$

Eliminate R from these equations and solve the resulting quadratic equation for I.

46. A number was to be added to ½ but by mistake ½ was divided by the number. However, the correct result was obtained. What was the number?

47. A tank can be filled by two pipes, but one takes 2 hours more than the other. If both pipes are open 1½ hr, the tank will be filled. How long does it take each pipe to fill the tank?

48. A metal ring has an outside radius of 10 in. and a width of x in. as shown in the diagram. What value of x will make the larger circle two times the area of the smaller one?



49. Solve the following simultaneous equations:

$$x^2 + y^2 = 16,$$
$$y - 3x = 2.$$

50. Solve the following simultaneous equations:

$$xy = 4,$$
$$y = x - 2.$$

27. Equations with radicals. In Chapter 3, we have learned to free algebraic expressions of radicals, and the same processes may be extended to equations that contain radicals. A radical equation is one that contains the unknown value under a radical sign. In most problems the indicated root is a square root. *The radical equation can be freed of the radical, as a general thing, by raising both members to the same power, which should correspond to the order of the radical.* If the resulting equation contains a radical, the squaring process is repeated.

EXAMPLE 4–39. Solve the equation $\sqrt{x + 11} = 4$.

Solution: $\sqrt{x + 11} = 4.$

Squaring, $x + 11 = 16.$

Then $x = 16 - 11 = 5.$

EXAMPLE 4–40. Solve the equation $\sqrt{x + 9} - \sqrt{x} = 1$.

Solution: It is best to transpose one of the radicals to the other member of the equation before squaring.

Thus, $\sqrt{x + 9} = 1 + \sqrt{x}.$

Now, squaring both members we get

$$(\sqrt{x + 9})^2 = (1 + \sqrt{x})^2$$

or $x + 9 = 1 + 2\sqrt{x} + x.$

Simplifying, $2\sqrt{x} = 8.$

Then $\sqrt{x} = 4$

and $x = 16.$

Check: $\sqrt{16 + 9} - \sqrt{16} = 1,$

$$5 - 4 = 1.$$

Care must be used in raising radicals to powers. When a separate square root is to be squared, it is accomplished by simply removing the radical sign. Thus, $(\sqrt{a})^2 = a$. But when the sum of two square roots is to be squared, it must be treated as any binomial. Thus, $(\sqrt{a} + \sqrt{b})^2$ is not equal to $a + b$, but rather is equal to $a + 2\sqrt{a}\sqrt{b} + b$. A close inspection of this form will show that it is in the general form of the square of a binomial. The middle term must be included.

Furthermore, the squaring of the radical expressions may result in introducing additional roots that do not satisfy the original equation. Therefore, all roots should be substituted back in the original equation to see if they satisfy. If they do not satisfy, they are *extraneous roots* and are discarded.

EXAMPLE 4–41. Solve the following equation for y:

$$\sqrt{1 + y\sqrt{y^2 + 12}} = 1 + y.$$

Solution: Squaring both members, $1 + y\sqrt{y^2 + 12} = 1 + 2y + y^2$. (Note that squaring the left member removes the radical sign only, since it covers the whole member.)

Then	$y\sqrt{y^2 + 12} = 2y + y^2.$
Squaring again,	$y^2(y^2 + 12) = 4y^2 + 4y^3 + y^4$
or	$y^4 + 12y^2 = 4y^2 + 4y^3 + y^4.$
Collecting terms,	$4y^3 - 8y^2 = 0.$
Factoring	$4y^2(y - 2) = 0.$
Then	$4y^2 = 0$ and $y - 2 = 0$
or	$y = 0$ or $y = 2.$

The roots of this equation, then, are 0 and 2. If we had divided through by y in the second step, the root $y = 0$ would have been lost. Therefore, an expression that contains the unknown should never be canceled out of an equation, because doing so may result in canceling out a root, or, what is more serious, dividing by zero.

EXERCISE 4-6

Solve the following radical equations:

1. $4\sqrt{x^2 - 9} + 4 = 4x.$
2. $\sqrt{14 + \sqrt{1 + \sqrt{x + 3}}} = 4.$
3. $2a + \sqrt{(3 - 5a)^2 + 16} = 5.$
4. $\sqrt{2E + \sqrt{10E + 1}} - \sqrt{2E} = 1.$
5. $\sqrt{3y - 5} + \sqrt{y - 9} - 2\sqrt{y - 1} = 0.$
6. $x - \dfrac{a^2}{\sqrt{x^2 - a^2}} + \sqrt{x^2 - a^2} = 0.$
7. $\sqrt{x} + \sqrt{x + 3b^2} = \sqrt{x + 8b^2}.$
8. $4\sqrt{4a^2 - 1} + 2a = 8a - \sqrt{4a^2 - 1}.$
9. $\sqrt{7 + \sqrt{1 + \sqrt{4 + \sqrt{1 + 2\sqrt{x}}}}} = 3.$
10. $\dfrac{\sqrt{x + a} + \sqrt{x - a}}{\sqrt{x + a} - \sqrt{x - a}} - \dfrac{\sqrt{x^2 - a^2}}{a} = 2.$
11. $\sqrt{6 + x} - 5 + \sqrt{2x} = 0.$
12. $\sqrt{3x} = 3\sqrt{3x - b} - \sqrt{4x + 6b}.$

28. Ratio and proportion. In Chapter 2 we found the meanings of ratio and proportion as applied to numbers. These meanings do not vary in any respect when letters are used. Thus we may have the ratio $a:b$ or the proportion $a:b = c:d$. The letters are used to make a general statement of any ratio or proportion.

It will be evident from the definition of a ratio that ratios have the same properties as fractions and may be treated as fractions. So we may:

1. Multiply or divide both terms of a ratio by the same number without changing the value of the ratio.

2. Multiply the ratio by any number if we multiply the numerator (or antecedent) by that number or divide the denominator (or consequent) by that number.

3. Divide the ratio by any number if we divide the numerator by that number or multiply the denominator by that number.

A proportion, from its definition, may be treated exactly as any equation because it is an equation. We have found that the product of the means of a proportion is equal to the product of the extremes. So in the proportion $a:b = c:d$ we know that $ad = bc$. This of course is an equation.

The following principles develop directly from the equation $ad = bc$:

$$(1)\ a = \frac{bc}{d}, \qquad (3)\ b = \frac{ad}{c},$$

$$(2)\ d = \frac{bc}{a}, \qquad (4)\ c = \frac{ad}{b},$$

Several other principles with respect to proportions have been developed. A few of them follow.

If four numbers are in proportion, they are in proportion by alternation.

If $\quad a:b = c:d,$
then $\quad a:c = b:d.$

If four numbers are in proportion, they are in proportion by inversion.

If $\quad a:b = c:d,$
then $\quad b:a = d:c.$

If four numbers are in proportion, they are in proportion by composition or addition.

If $\quad a:b = c:d,$
then $\quad a + b:b = c + d:d;$
also $\quad a + b:a = c + d:c.$

If four numbers are in proportion, they are in proportion by division or subtraction.

If $\quad a:b = c:d,$
then $\quad a - b:b = c - d:d,$
or $\quad a - b:a = c - d:c.$

If four numbers are in proportion, they are in proportion by composition or addition, and division or subtraction.

If $$a:b = c:d,$$
then $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

If four numbers are in proportion, their like powers and like roots are in proportion.

If $$a:b = c:d,$$
then $$a^n:b^n = c^n:d^n,$$
and $$\sqrt[n]{a}:\sqrt[n]{b} = \sqrt[n]{c}:\sqrt[n]{d}.$$

29. Variation. The idea of variation is a common one. Since everything is affected by its environment, it is logical to think of things changing as other things around them change. For instance, the amount of pay a man receives varies directly with the time he works; that is, the longer he works, the more pay he will receive. Again, the resistance of a copper wire varies directly with the length and inversely with its cross-sectional area; that is, as the length is increased, the resistance will also increase; but as the cross-sectional area is increased, the resistance will decrease.

In variation, we have to deal with certain numbers called *variables* and other numbers called *constants. A variable is a number whose value may change, whereas a constant is a number whose value remains unchanged.* A variable is generally denoted by the last letters of the alphabet, x, y, and z, or by the first letter in the name of the quantity such as l for length or R for resistance. A constant is generally denoted by the first letters of the alphabet, a, b, and c, or by k or by letters of the Greek alphabet. The symbol denoting a variation is $\propto$ and is read "varies as."

In the first example cited, if x represents the pay a man receives and y represents the time he works, the variation may be written $x \propto y$, which is merely a mathematical statement of the word statement that the pay varies directly as the length of time worked.

Again, the resistance of a copper conductor varying directly with the length is written $R \propto l$, and the resistance varying inversely with the cross-section area is written $R \propto \frac{1}{A}$.

A statement of variation is not an equation and must not be considered as such. However, it can be made into an equation by the use of the proper constant. Thus, the variation $x \propto y$, that we have already shown as an example, can be changed to an equation by using a constant K. This constant would be the pay received per day if y is equal to the number of days worked. The equation then would read $x = Ky$. In like manner, the variation $R \propto l$ is made into an equation by using a constant that represents the resistance per foot of the conductor. The equation would be $R = Kl$. The variation $R \propto \frac{1}{A}$ is converted into an equation by

using a constant that represents the resistance of a unit area. The equation is $R = \frac{K}{A}$.

If a quantity varies jointly as two other quantities, then it varies as their product. Thus, if x varies jointly with y and z, the equation would be $x = Kyz$ where K has the necessary value to make an equation.

If a quantity varies directly with one quantity and inversely with another, then it varies as the quotient of the first divided by the second. This is the case with the resistance of a copper conductor which varies directly with the length and inversely with the cross-sectional area. The equation would then be $R = \frac{Kl}{A}$, in which K would have the proper value for a unit length and a unit cross section. This equation is generally written as $R = \frac{\rho l}{A}$, where ρ represents the resistance of a circular mil foot of the material.

In the solution of problems the constant very often can be eliminated by forming a proportion out of two equations that contain the same constant. For instance, if R_1 represents the resistance of a copper conductor of length l_1 and cross section A_1, and R_2 represents the resistance of another copper conductor of length l_2 and cross section A_2, the two equations would contain the same constant ρ, and would read

$$R_1 = \frac{\rho l_1}{A_1}$$

and

$$R_2 = \frac{\rho l_2}{A_2}.$$

By dividing the first equation by the second we get a new equation from which the constant has been eliminated. Thus,

$$\frac{R_1 = \frac{\rho l_1}{A_1}}{R_2 = \frac{\rho l_2}{A_2}},$$

or

$$\frac{R_1}{R_2} = \frac{l_1 A_2}{l_2 A_1}.$$

Now, if R_2 is the only unknown value in this equation, it can be solved for R_2 in terms of the other letters. Thus,

$$R_2 = \frac{R_1 l_2 A_1}{l_1 A_2}.$$

An inspection of this equation will show that R_2 is directly proportional to each of the quantities R_1, l_2, and A_1 and is inversely proportional to each of the quantities l_1 and A_2. However, the direct or inverse proportionality would not necessarily hold if there were sums or differences of

variable quantities involved. Thus, in the equation $t = \frac{T(C - F)}{D}$, t is directly proportional to T and to the quantity $C - F$ and inversely proportional to D, but t is not directly proportional to either C or F alone unless both are changing at exactly the same rate, or unless one of them is a constant value.

EXAMPLE 4–42. The resistance of a coil of copper wire $\frac{3}{16}$ in. in diameter is 2.5 ohms. What is the resistance of a copper wire of the same length with the diameter of $\frac{3}{8}$ in.?

Solution: The two equations are

$$R_1 = \frac{\rho l_1}{d_1^2} \quad \text{and} \quad R_2 = \frac{\rho l_2}{d_2^2}.$$

By division

$$\frac{R_1}{R_2} = \frac{\dfrac{\rho l_1}{d_1^2}}{\dfrac{\rho l_2}{d_2^2}} = \frac{l_1 d_2^2}{l_2 d_1^2}.$$

Since l_1 and l_2 are equal, they cancel out and the equation solved for R_2 becomes

$$R_2 = \frac{R_1 d_1^2}{d_2^2},$$

$$R_2 = \frac{(2.5)\left(\dfrac{3}{16}\right)^2}{\left(\dfrac{3}{8}\right)^2} = (2.5)\frac{\left(\dfrac{3}{16}\right)^2}{\left(\dfrac{3}{8}\right)^2} = (2.5)\left(\frac{1}{2}\right)^2 = 0.625 \text{ ohm}.$$

EXERCISE 4–7

1. The volume of a sphere varies directly as the cube of the radius. If a sphere whose radius is 4 in. contains 268.07 cu in., find the volume of a sphere whose radius is 6 in.

2. The illumination from a source of light varies inversely as the square of the distance from the light. If a book now 15 in. away from the light is moved to 20 in. distance, what is the ratio of the new illumination to the old?

3. The number of oscillations of a pendulum per time unit varies inversely as the square root of its length. The length of a pendulum that makes one oscillation per second at the sea level is 39.1 in. How long is a pendulum that oscillates 3 times a second?

4. The safe load of a horizontal beam supported at both ends varies as the breadth and the square of the depth, and inversely as the length between the supports. If a white pine joist 6 in. wide, 2 in. deep, and 10 ft long safely holds 800 lb, what is a safe load for a joist of the same material 8 in. wide, 4 in. deep, and 12 ft long?

5. The pressure of wind on a sail varies jointly as the area of the sail and the square of the wind's velocity. When the velocity of the wind is 18 mph, the pressure on a square foot is 1 lb. What is the velocity of the wind when the pressure is 4 lb?

6. The surface of a cube varies as the square of its edge. If the surface of a cube, whose edge is $2\frac{1}{3}$ in., is $32\frac{2}{3}$ sq in., what will be the edge of a cube whose surface is $30\frac{3}{8}$ sq in.?

7. The illumination from the source of light varies inversely as the square of the distance from the light. If a book now 12 in. away from the light is moved to 24 in. distance, what is the ratio of the new illumination to the old?

8. The resistance of a coil of copper wire 0.2043 in. in diameter is 1.313 ohms. What is the resistance of a copper wire of the same length that has a diameter of 0.1620 in.?

REVIEW EXERCISE 4-8

Solve the following linear equations for values of the unknown:

1. $12x - 5 = 2x + 15.$
2. $4y + 7 = 8y - 17.$
3. $13 - 3x = 41 - 10x.$
4. $2(6 - 2y) + 3(1 - y) = 4(5 - 2y).$
5. $2x = \frac{5}{6} + \frac{x}{2}$
6. $y - \frac{y}{7} - 4 = 0.$
7. $\frac{y}{4} - 5 - \frac{y}{6} = 3.$
8. $0.5I - 0.575 = \frac{I}{5} + \frac{7}{40}.$
9. $2x - 16 - 3[2x - 4(x - 2)] = 4x.$
10. $\frac{6x - 4}{5} + 2\frac{3}{4} = \frac{4x + 8}{2} - 5\frac{1}{4}.$
11. $5 + y = 15y - [-3 - (y - 1) + 12].$
12. $(x^2 - 2) = (x - 2)^2.$
13. $(x + 4)^2 = (x - 4)^2 + 16.$
14. $(3x - 2)^2 = 9(x + 2)^2.$
15. $4(2x + 5) = 2(4x - 7).$
16. $\frac{E}{2} + \frac{E}{3} - \frac{E}{4} - \frac{3E}{10} + \frac{5E}{12} = 3\frac{1}{2}.$
17. $\frac{4x - 2}{6} - \frac{3x - 4}{4} = \frac{2x + 2}{3} - \frac{7x - 2}{9}.$
18. $y - \frac{20 - y}{2} + \frac{2(3y - 4)}{7} = \frac{7(y - 1)}{5} - 4.$
19. $\frac{7y}{12} + \frac{11y - 2}{9} = \frac{2(y + 4)}{3} + \frac{y}{36} + \frac{1}{6}.$
20. $\dfrac{x - \frac{3}{2}}{x - 1} = \dfrac{x - \frac{1}{2}}{x + \frac{1}{2}}$
21. $(x - a)(x + b) - (x + a)(x - b) = 2.$

22. Solve Problem 2 of Exercise 4–2 for b; for h.
23. Solve Problem 4 of Exercise 4–2 for h.
24. Solve Problem 5 of Exercise 4–2 for R.
25. Solve Problem 6 of Exercise 4–2 for N.
26. Solve Problem 10 of Exercise 4–2 for g.
27. Solve Problem 11 of Exercise 4–2 for V.
28. Solve Problem 12 of Exercise 4–2 for a.
29. Solve Problem 13 of Exercise 4–2 for R_1; for R_2.
30. Solve Problem 14 of Exercise 4–2 for R_1; for R_2; for R_3.
31. Solve Problem 15 of Exercise 4–2 for N; for A; for l.
32. Solve Problem 16 of Exercise 4–2 for f; for l.
33. Solve Problem 17 of Exercise 4–2 for f; for c.
34. Solve Problem 18 of Exercise 4–2 for N; for S.
35. Solve Problem 19 of Exercise 4–2 for I_a; for R_a; for ϕ.
36. Solve Problem 20 of Exercise 4–2 for q'; for r.
37. Solve Problem 21 of Exercise 4–2 for I_p; for I_s; for N_p.
38. Solve Problem 22 of Exercise 4–2 for p.
39. Solve Problem 23 of Exercise 4–2 for N; for l; for A; for μ.
40. Solve Problem 24 of Exercise 4–2 for I.
41. Solve Problem 25 of Exercise 4–2 for R.
42. Solve Problem 26 of Exercise 4–2 for E_b; for i.
43. Solve Problem 27 of Exercise 4–2 for R_1; for R_2; for R_3.
44. Solve Problem 28 of Exercise 4–2 for P; for B; for y.
45. Solve Problem 29 of Exercise 4–2 for D.
46. Solve Problem 30 of Exercise 4–2 for C; for F; for D.
47. Solve Problem 31 of Exercise 4–2 for I.

48. The following equation is the formula for the factor (K) giving the R.F. resistance of a copper wire in terms of its ohmic resistance and holds true for factors greater than 8:

$$K = 0.25 + 0.0962d\sqrt{f},$$

where d = diameter of wire in inches,
f = cycles per second.

Solve the formula for f and determine the frequency in megacycles required for: (a) a factor of 10 and a No. 10 wire; (b) a factor of 10 and a No. 30 wire.

Note: 1 megacycle = 1,000,000 cycles,
dia. of No. 10 wire = 0.1 in. approx.,
dia. of No. 30 wire = .01 in. approx.

49. Solve the equation of Problem 48 for d and determine the size wire required to give a factor of 12 and a frequency of 10 megacycles.

50. The perimeter of a triangle is 75 in. If the second side is 6 in. longer than the first and 3 in. shorter than the third, what is the length of each side?

51. A number consists of two digits whose sum is 7. If the digits are reversed, the new number is 27 more than the given number. What is the given number?

Hint: A number consisting of two digits, x and y, is not expressed as xy but as $10x + y$ where y is the digit in the unit's place and x is the digit in the ten's place.

52. Find three consecutive numbers whose sum is 111.

53. A beam 24 ft long weighs 450 lbs and is balanced on a fulcrum by a load of 150 lbs attached to one end. How far from the load end must the fulcrum be placed?

Note: The weight of the beam may be considered as concentrated at the center and the load multiplied by its distance to the fulcrum must equal the weight of the beam multiplied by its distance to the fulcrum to obtain equilibrium.

54. A party travels 460 miles averaging 40 mph for part of the journey and 30 mph for the rest. If it took 13 hours to complete the journey, how far did the party travel at each rate?

55. A reservoir is filled by one pipe in 40 minutes or by another pipe in 30 minutes. How long will it take to fill the reservoir if both pipes are operating at the same time? A third pipe can empty the reservoir in 20 minutes. If all three pipes are open how long will it take to fill the reservoir?

56. By use of the equation in Problem 24 of Exercise 4–2, determine the current drawn by an electric iron that produces 1200 watts of power (P) when it has a resistance of 12 ohms.

57. A Wheatstone bridge testing set is used to measure an unknown resistance. When the bridge is balanced, $R_3 = 7538$, and the ratio of R_2 to R_1 is 1 to 100. Determine the value of the unknown. Use equation of Problem 27, Exercise 4–2.

58. A series resonant circuit in an alternating current circuit occurs when the inductive reactance (x_l) is equal to the capacitive reactance (x_c). If $x_l = 2\pi fL$ and $x_c = \frac{10^6}{2\pi fc}$, determine the equation for the resonant frequency, f. What will be the value of f when $L = .05$ and $c = 2.5$?

59. A has \$1200 and B has \$700. After A spends half again as much as B spends, he has three times as much left as B. How much has each spent?

60. We desire to obtain 10 gallons of anti-freeze mixture which is 50% alcohol. We have 10 gallons of a 35% mixture and 12 gallons of a 75% mixture. How much of each shall we use?

What percentage solution would we have by mixing all of the 35% with all of the 75%?

61. From the equation of Problem 14, Exercise 4–2, develop the expression for R_t and determine R_t when $R_1 = 10$, $R_2 = 15$, and $R_3 = 18$.

62. We have two grades of iron ore, the first containing 45% zinc

and the second 25% zinc. How many pounds of each are required to make a mixture of 1000 lbs containing 40% zinc?

63. A transcontinental air flight required 11 hours in one direction with the prevailing winds whereas it took an hour longer in the opposite direction. If the normal plane speed in still air is 250 mph, what is the average wind velocity?

64. When shooting at a target, the sound of the bullet is heard striking the target 3 sec after the bullet is fired. If the speed of the bullet is 2500 ft per sec and sound travels at the rate of 1100 ft per sec, how far away is the target?

65. Sound travels in air at a speed in feet per second determined by the equation

$$V = 1090 + 1.14(\mathrm{F} - 32).$$

where F is the temperature in Fahrenheit degrees. What is the temperature when the speed is 1100 ft per sec?

66. Convert the equation of Problem 65 for use with centigrade degrees. Determine the temperature of Problem 65 in degrees centigrade by this new equation.

67. Two resistances, R_1 and R_2 in parallel, give a total resistance, R_t, according to the following equation:

$$R_t = \frac{R_1R_2}{R_1 + R_2}.$$

If a total resistance of 20 ohms (R_t) is required, what resistance, R_2, should be placed in parallel with one of 50 ohms (R_1) to obtain this result?

68. The following empirical equation gives the value of the force (F) in pounds necessary to move a flat plate area (A) in sq ft with a velocity (V) in feet per sec perpendicular to an air stream:

$$F = 1.52 \times 10^{-3}AV^2.$$

What will be the equation if the velocity is in miles per hour? Determine the force pressing against an automobile windshield 48 in. by 16 in. if the car is traveling at a speed of 60 mph.

69. What HP will be used up by the car in Problem 68 in overcoming the wind resistance?

$$\mathrm{HP} = \frac{(F \text{ in lbs})(V \text{ in ft per min})}{33{,}000}.$$

What will be the HP equation with A in sq in. and V in ft per sec? With V in miles per hour? Derive each by use of the equation in Problem 68.

70. Power in watts in a d.c. electrical circuit is equal to the product of the voltage and current or $P = EI$. Derive the equation for P in terms of E and R by use of Ohm's law, $E = IR$.

What will be the resistance of a 60 watt lamp placed on 120 volts?

Solve the following sets of simultaneous equations:

71. $x + 2y = 17,$
$2x - y = -1.$

72. $\dfrac{y}{x+1} = 3,$
$\dfrac{y+1}{x} = 4.$

73. $0.6x - 0.8y = 0.5,$
$0.9x + 0.7y = 1.7.$

74. $\dfrac{x}{4} + 8 = \dfrac{y}{2} - 12,$
$\dfrac{x+y}{5} - \dfrac{2x-y}{4} - 8 = 27 - \dfrac{y}{3}.$

75. $rx + sy = r + s,$
$r^2x + s^2y = 1.$

76. $5x - 2y - 3z = 9,$
$3x - 5y + 4z = 38,$
$7x + 3y - 5z = -13.$

77. In Figure 4–1, page 144, let $E_1 = 18$, $E_2 = 15$, $R_1 = 0.3$ ohm, $R_2 = 0.15$ ohm, $R_3 = 3.5$ ohms, $R_4 = 5$ ohms, $R_5 = 2.4$ ohms, $R_6 = 4.2$ ohms, $R_7 = 2.2$ ohms. Solve the three simultaneous equations formed for I_1, I_2 and I_3.

78. $\dfrac{4}{x} - \dfrac{3}{y} + \dfrac{12}{z} = 3,$
$-\dfrac{5}{x} - \dfrac{7}{y} + \dfrac{8}{z} = 10\dfrac{2}{3},$
$\dfrac{3}{x} + \dfrac{6}{y} - \dfrac{1.6}{z} = 0.$

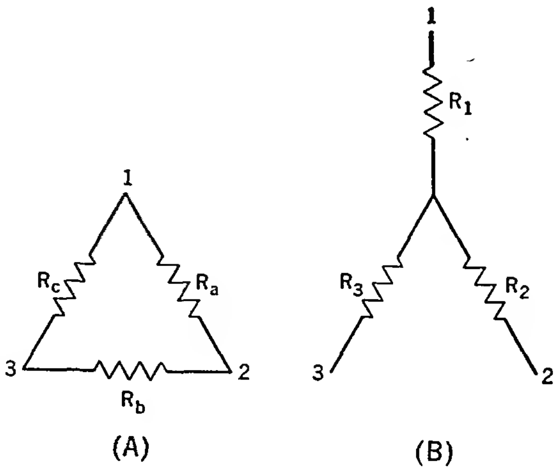
79. $a + b + c + d = 0,$
$5a + 3b - 4c - 2d = 7,$
$3a + 2b + 3c + 4d = 8,$
$4a + 5b - 6c - 5d = -8.$

80. $x + y + 2z = 3,\ 2y - 4z + 5w = 5,\ 3x - 5y = 1,\ 5y + 2w = 0.$

81. As a supplement to Kirchhoff's laws in the solution of complex circuit problems, it is convenient at times to convert from a delta connection (A) to an equivalent wye connection (B). This means that the resistance between terminals 1 and 2, 2 and 3, and 3 and 1 shall be the same for each circuit.

To make these circuits equivalent, the following equations must be true:

$$R_1 + R_2 = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c},$$
$$R_2 + R_3 = \frac{R_b(R_a + R_c)}{R_a + R_b + R_c},$$
$$R_3 + R_1 = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c}.$$



Solve these equations for R_1, R_2, and R_3 in terms of R_a, R_b, and R_c. Determine the equivalent wye resistances for delta resistances of $R_a = 4$ ohms, $R_b = 5$ ohms, and $R_c = 6$ ohms.

82. The two equations, $S = 0.2619DN$ and $T = \frac{LF}{N}$, occur in lathe work where

S = cutting speed in feet per minute,
D = diameter of work in inches,
L = length of work in inches,
N = rpm,
T = time in seconds to do the work,
F = number of turns required for a side movement of 1 in.

Eliminate N from the equations and solve for the value of T in terms of the remaining letters.

Find the time required to turn out a piece when $D = 2.5$ in., $S = 12$ ft/min, $L = 24$ in. and $F = 20$.

Solve the following quadratic equations. (Check the answers to Problems 83 to 88 inclusive by the sum and product of the roots.)

83. $a^2x^2 + (c - d)ax = cd.$

84. $2m^2x^2 - 7mnx + 6n^2 = 0.$

85. $\dfrac{1}{\frac{x}{3} - 1} + \dfrac{3}{\frac{x}{2} - 1} - \dfrac{2}{\frac{x}{4} - \frac{7}{10}} = 0.$

86. $\dfrac{R - 5}{R + 2} = \dfrac{67}{4(R^2 - 4)} - \dfrac{2}{R - 2}.$

87. $.04(x - 4)(x + 3) - .02(x - 1) = .03(x + 1) + .006(x + 5).$

88. $\dfrac{3E - 2}{E} - \dfrac{E}{E - 4} = \dfrac{3}{4}.$

89. What value must m have in the equation $(mx)^2 + 24x + 16$ so that the roots will be real and equal?

90. In the equation $(3x)^2 - 42x + n = 0$, what value must n have to make the roots real and equal? What values of n will make the roots real but unequal? What values of n will make the roots imaginary?

91. The percent mark-up on a piece of equipment was half as large as the cost price in dollars. The piece sold for \$42.48. How much did it cost?

92. After traveling 75 miles at a certain average speed in a car, a salesman suddenly realized that his average speed would have to be increased 5 mph in order to make the total distance of 110 miles on time. Had he kept traveling at the original rate, he would have been 10 minutes late for his appointment. How long did it take him to make the trip?

93. If a body is thrown vertically downward from a height of x feet with an initial velocity of V_0 ft per sec, its height at the end of t sec can be found by means of the equation $h = x - V_0t - 16t^2$. How long will it take a body to reach a height of 400 feet from the ground if it is thrown downward from a height of 2,000 ft with an initial velocity of 32 ft per sec? How long will it take to reach the ground from the original height?

94. A plane was driven 450 miles at a certain speed. The speed was then increased by 30 miles per hour, and the plane continued on its course, landing after a total elapsed time of $4\frac{1}{2}$ hours. The total distance covered was 720 miles. What were the two flying speeds?

95. The surface area of a cylinder is given by the equation $S = 2\pi r^2 + 2\pi rh$. Solve the equation for r and determine the radius of a cylinder with a height of 4 in. and a surface area of 120 sq in.

Solve the following radical equations:

96. $2\sqrt{y} - \sqrt{y-3} = \sqrt{2y+1}$.

97. $\dfrac{\sqrt{x}}{x-b} = \dfrac{b}{\sqrt{x}}$.

98. $x - b = \sqrt{b^2 + x\sqrt{a^2 + x^2}}$.

99. $\sqrt{9 + 2x\sqrt{2x-15}} + 3 = x$.

100. $2\sqrt{x-2} + \sqrt{x+9} = \sqrt{2}\sqrt{5x-7}$.

101. $\dfrac{\sqrt{4x}+3}{6+\sqrt{x}} = \dfrac{6-\sqrt{x}}{\sqrt{x}}$.

102. $\dfrac{5\sqrt{x}-18}{3\sqrt{x}-6} = \dfrac{7\sqrt{x}-14}{6\sqrt{x}-8}$.

103. $2\sqrt{x+6} - \dfrac{12}{\sqrt{x+6}} = \sqrt{x+15}$.

104. $\sqrt{x^2+x+7} - \sqrt{x^2-x-5} = 2$.

105. $2\sqrt{20x} - \sqrt{6x-5} - 15 = 0$.

Variation

106. The horsepower required to drive a ship through still water varies with the cube of the ship's speed. If it takes 1,200 horsepower for a speed of 10 knots, what horsepower will be necessary for a speed of 22 knots?

107. The period of vibration of a pendulum varies directly with the square root of its length. If a pendulum 36 in. long makes one oscillation per second, what will be the period of vibration of a pendulum that is 12 in. long?

108. A book is 18 in. from a source of illumination. How far toward the source will it have to be moved to make the illumination twice as much? The illumination varies inversely as the square of the distance from the source.

109. Determine the radius of a circle whose area will be doubled if the radius is increased by 2 in.

110. The power loss in a direct current circuit is equal to the product of the resistance and the square of the current, or $P = RI^2$. If the resistance is doubled and the current is halved, what will be the effect on the power loss?

111. What will be the percentage increase in the volume of a sphere if the diameter is increased by 10%?

112. The resistance of an electrical conductor varies directly with its length and inversely with its cross sectional area. What will be the percentage change in resistance if the diameter is increased 25% and the length decreased 10%?

113. The volume of a gas at constant temperature varies inversely with the pressure applied. If 3 cu ft of gas is forced by a pressure of 25 lbs per sq in. into a vacuum tank that will hold 10 cu ft, what will be its pressure after it has expanded to fill the tank?

114. The horsepower required to drive an airplane is directly proportional to the cube of its speed. If 1,200 horsepower will maintain a speed of 125 mph, what horsepower will be required for 300 mph?

115. In Problem 114, what speed will be developed by a horsepower of 12,000?

Formulas

Solve the following formulas for the letters indicated:

	Given:	*Solve for:*
116.	$R = R_g + \dfrac{R_c}{\dfrac{R_p + R_c}{R_p}};$	R_p; R_c.
117.	$A_v = \dfrac{\mu M}{(R_s r_p + \omega^2 M^2)C};$	M; C; μ.
118.	$A_v = \dfrac{\mu R_L}{r_p + R_L};$	R_L; μ.
119.	$E_b = K\dfrac{E_p}{\mu} + E_g;$	E_g; μ.
120.	$e_z = \dfrac{\mu e_g Z_2}{r_p\left(\dfrac{Z_1 + R_2}{Z_1}\right) + R_2};$	Z_1; R_2; μ.
121.	$R_b = \dfrac{E_2 - E_1}{I_2 - I_1};$	E_2; I_1.
122.	$P = \dfrac{R(r^2 + x^2)}{r(Rr + Xx)};$	x; r.
123.	$b = \dfrac{X}{R^2 + X^2};$	R; X.
124.	$c = 2\pi\sqrt{\dfrac{a^2 + b^2}{2}};$	a; b.
125.	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1;$	x; y.

Chapter 5

GRAPHICAL REPRESENTATION

1. The graph. In a laboratory experiment on a direct-current generator, the generator was run at constant speed without load and the following readings of voltage and field current were obtained:

Voltage	*Field current*	*Voltage*	*Field current*
60	0.075	275	0.39
125	0.14	300	0.45
150	0.175	330	0.55
170	0.20	350	0.60
200	0.25	380	0.80
220	0.30	400	0.95
250	0.35		

The variation in voltage with respect to current is evident from these figures, but it may be seen more easily if we put the information into a diagram as shown in Fig. 5–1.

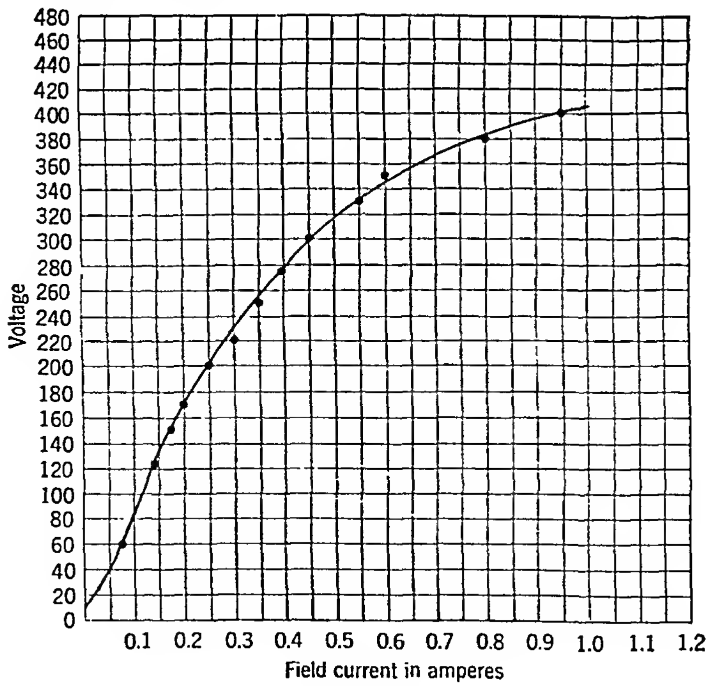


Fig. 5–1.

Here the values of field current are located on the horizontal line and the values of voltage are located on the vertical line. The voltage for any value of field current is located so as to be directly above the corresponding current. In this manner we have located 13 points and have drawn a curve that represents the average of the points. This method of so representing relations between numbers is called a *graphical method*, and the curve drawn through the points is called a *graph*. The making of the graph is generally spoken of as *plotting*.

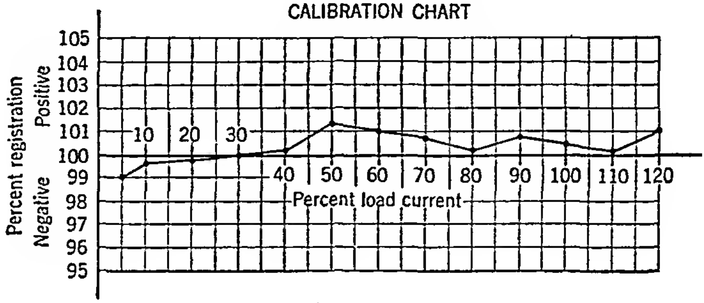


Fig. 5-2.

It should be noted that, owing to inaccuracies in testing, every point may not be exactly on the curve. Hence, when it is assumed that testing errors exist, a smooth curve is drawn which will include as many points as possible and will make the other points fall on either side of the curve. In calibration tests, however, it is quite generally assumed that the test is accurate and that the apparatus being calibrated is subject to error. In such cases a straight line from point to point is used. See Fig. 5-2.

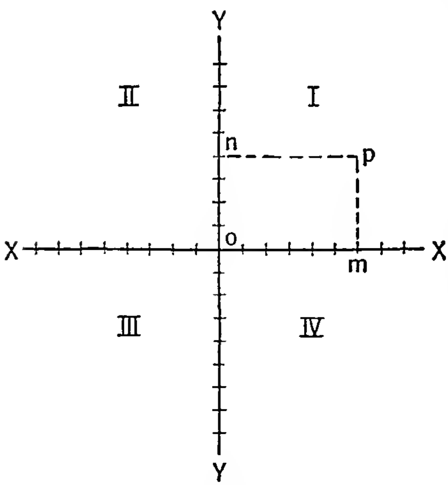


Fig. 5-3.

2. Rectangular coördinates. The graph of voltage plotted against field current shows only values upward for voltage and values to the right for field current. However, it is oftentimes necessary to plot values downward and to the left and so it is necessary to extend our treatment to include such values.

If, as in Fig. 5-3, two lines $x - x$ and $y - y$ are drawn at right angles to each other, the position of any point, such as P, may be determined by

measuring its distance from each of these lines. These distances, Om and On, are called the *coördinates* of the point P.

The two lines $x - x$ and $y - y$ are called the coördinate axes, $x - x$ being called the x-axis or *axis of abscissas* and $y - y$ being called the y-axis or *axis of ordinates*. The coördinates, Om and On, of the point P are the *x-coördinate* and the *y-coördinate* respectively. The point where the axes meet, denoted by O, is called the *origin*. The coördinates of any point are always measured from the axis. Positive values for the x-coördinate are plotted to the right along the x-axis and negative values for the x-coördinate are plotted to the left along the x-axis. Positive values for the y-coördinate are plotted upward along the y-axis and negative values are plotted downward along the y-axis.

It may be seen from an inspection of the figure that the x- and y-axes divide the plane on which they are constructed into four parts. These parts are called *quadrants* and are numbered I, II, III, and IV, as shown on the figure.

Since all positive values of the x-coördinate are plotted to the right, with negative values to the left, and positive values of the y-coördinate upward, with negative values downward, the following statements should be evident:

In quadrant I: *x and y are both positive.*
In quadrant II: *x is negative, y is positive.*
In quadrant III: *x and y are both negative.*
In quadrant IV: *x is positive, y is negative.*

In order to plot a point, we must first draw the x- and y-axes and then pick some convenient scale for the x values and for the y values. Then the x-coördinate is laid off on the x-axis and the y-coördinate laid off on the y-axis. Finally, perpendicular lines are erected at the ends of these coördinates, and the junction point of these perpendiculars gives the desired point. The coördinates of a point are always given with the x value first; thus, $(+4, -3)$ means an x-coördinate of $+4$ and a y-coördinate of -3.

EXAMPLE 5–1. Plot the following points: $(2, 4)$, $(-5, 3)$, $(-5, -4)$, $(6, -2)$, and find the distance of each point from the origin.

Solution: The solution may be seen in Fig. 5–4. Each square represents one unit for x and one unit for y.

To plot the point $(2, 4)$, start at the origin and lay off 2 units to the right on the x-axis. Then starting at the origin again, lay off 4 units upward along the y-axis. Next erect perpendiculars to the x- and y-axes where the respective coördinates (2 and 4) terminate. The intersection of these perpendiculars will give the desired point as shown in the figure.

To plot the point $(-5, 3)$, lay off 2 units negatively on the x-axis and 3 units positively on the y-axis and erect perpendiculars to the axes. The intersection of these perpendiculars will give the desired point.

The other points $(-5, -4)$ and $(6, -2)$ are plotted in a similar way.

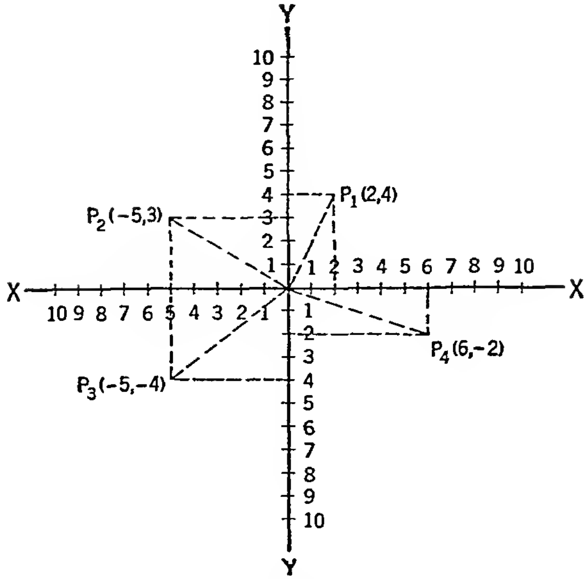


Fig. 5-4.

Since the distance from the origin to the point forms the hypotenuse of a right triangle whose legs are the x- and y-coördinates respectively, the right triangle formula may be used. Therefore, the distances from the origin to each point may be found by taking the square root of the sum of the squares of the x- and y-coördinates. This distance is always considered positive. Thus,

$$OP_1 = \sqrt{(2)^2 + (4)^2} = \sqrt{20} = 4.47,$$
$$OP_2 = \sqrt{(-5)^2 + (3)^2} = \sqrt{34} = 5.82,$$
$$OP_3 = \sqrt{(-5)^2 + (-4)^2} = \sqrt{41} = 6.40,$$
$$OP_4 = \sqrt{(6)^2 + (-2)^2} = \sqrt{40} = 6.32.$$

3. Functions and graphs of equations. If two quantities, such as x and y, are so related to each other that when a value is given to x, one or more values are determined for y, then y is said to be a *function* of x. This means that y depends upon x for its values, or, as x changes in value then y must also change in value. Hence, x and y are called *variables*. In the expression $y = 3x + 2$, values may be substituted for x and the corresponding values for y determined. These values may be put down in tabular form:

x	-4	-3	-2	-1	0	1	2	3	4
y	-10	-7	-4	-1	2	5	8	11	14

Now we can plot these values on our rectangular coördinate scales and, by joining all the points, construct the graph of the function.

The variable x, whose values are arbitrarily assumed, is called the *independent variable* and is plotted along the x-axis. The variable y, whose values depend upon the values chosen for x, is called the *dependent variable* and is plotted along the y-axis.

The relation expressed by the statement that y is a function of x is generally written in the abbreviated form $y = (f)x$. The graph of a function and the graph of an equation have the same meaning. However, it may be necessary to solve an equation for y in order to put it in the functional form. Thus the equation $3x + 2y = 8$ solved for y would give $y = \frac{8 - 3x}{2}$. Both, however, are represented by the same graph.

Any equation that contains only the first degree of the variables is a *linear equation*, and a linear equation with two variables forms a straight line in one plane. Such an equation when plotted will have an infinite number of solutions and is therefore called an *indeterminate equation*, but any two points that satisfy the equation are sufficient to determine its graph. The straight line will extend through two such points in either direction indefinitely. The two points most frequently chosen are those points where $x = 0$ and where $y = 0$. These points are called the *intercepts* since they come at the places where the graph intersects each axis.

Other equations with powers higher than the first will not be straight lines but will be curves whose contours are determined by the form of the equation. For instance, an equation with the second power of one variable and the first power of the other variable represents a *parabola*. Thus, $x^2 = 8y$ is a parabola. Again, an equation with the second power of both variables may be a *circle* or an *ellipse*. Thus, $x^2 + y^2 = 25$ is a circle, whereas $9x^2 + 4y^2 = 36$ is an ellipse.

However, the graph of any kind of equation in two variables can be plotted by substituting values for one of the variables and determining the corresponding values for the other variable. Several sets of values are found in this way and then plotted to a suitable scale on graph paper. *It is not necessary that the scales chosen be the same for the x-axis and the y-axis.* The scales should be so chosen that the plotting can be done easily and quickly. Examples 5–2, 5–3, and 5–4 illustrate the method.

EXAMPLE 5–2. Plot the graph of the equation $3x + 2y = 8$.

Solution: Tabulating values of y with assumed value of x, we get

x	-4	-3	-2	-1	0	1	2	$2\frac{2}{3}$	3	4
y	10	$8\frac{1}{2}$	7	$5\frac{1}{2}$	4	$2\frac{1}{2}$	1	0	$-\frac{1}{2}$	-2

Any two sets of these points will determine the line. Therefore, by choosing a scale of one square equal to one unit for x or one unit for y and plotting, we get the graph of Fig. 5–5.

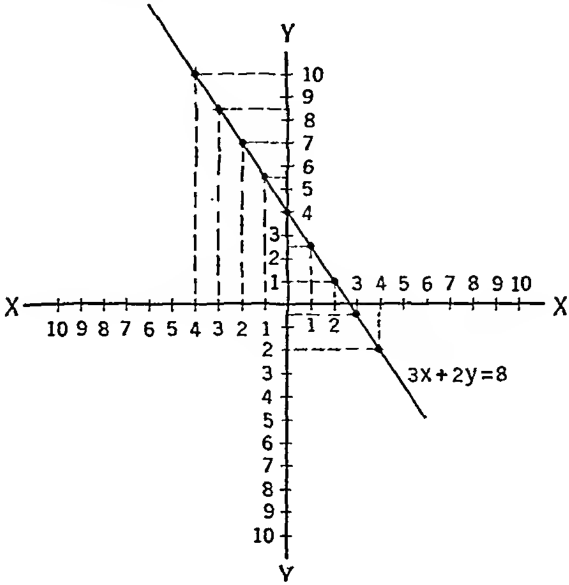


Fig. 5–5.

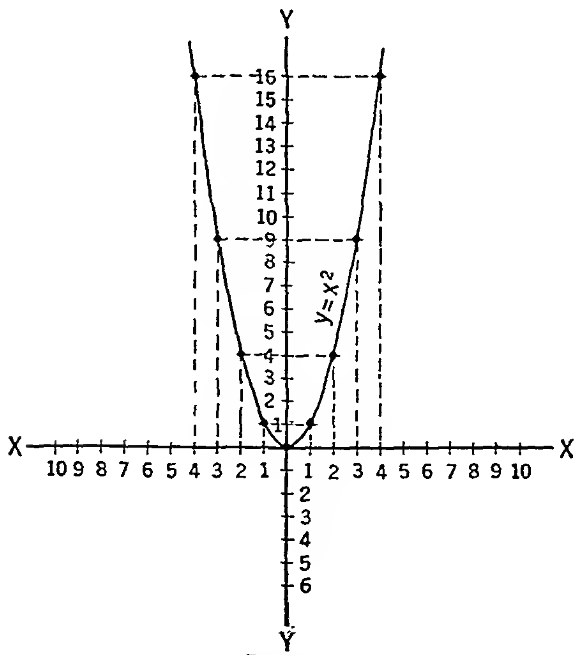


Fig. 5–6.

EXAMPLE 5–3. Plot the graph of the equation $y = x^2$.

Solution: Tabulating x and y values,

x	-4	-3	-2	-1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

Choosing a suitable scale and again plotting, we have the graph of Fig. 5–6.

EXAMPLE 5–4. Plot the graph of the equation $y = \dfrac{1}{x}$.

Solution: Substituting values for x and finding the corresponding value of y, we get

x	-8	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	∞	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

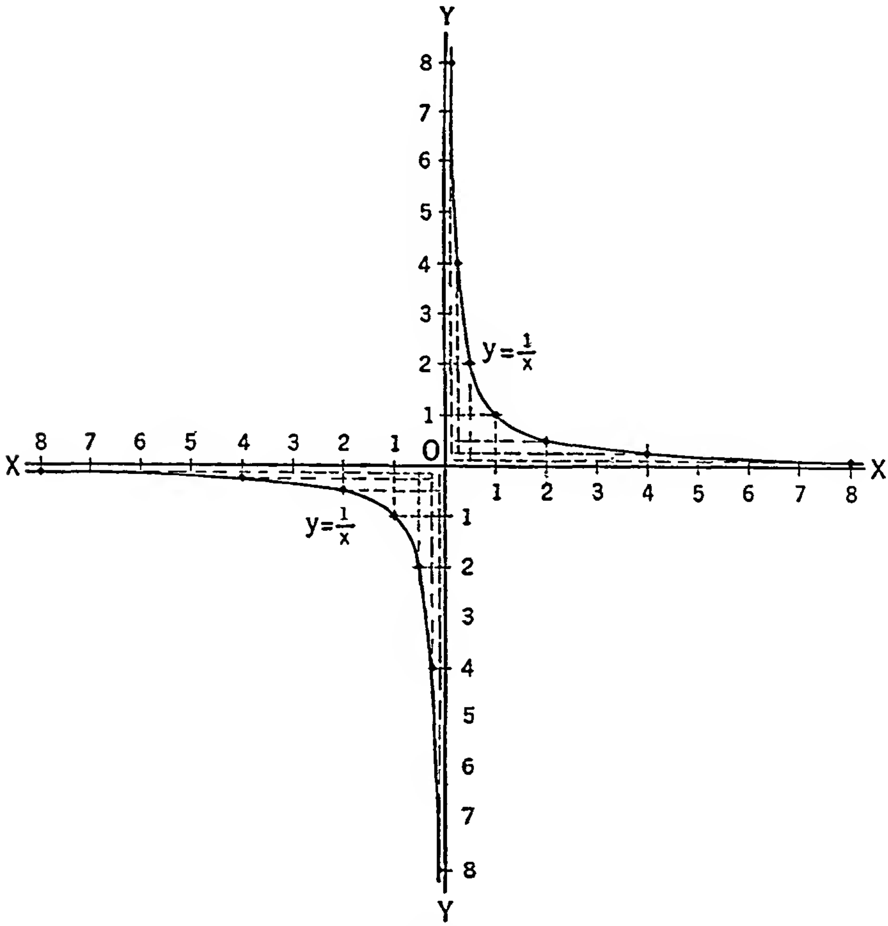


Fig. 5–7.

It may be seen from an inspection of this graph and the values plotted thereon that as x increases in size, positively or negatively, y will decrease in size, positively or negatively, but neither x nor y can be zero, because the other value would then become infinitely large and could not be plotted. This is the graph of a *hyperbola*.

4. Roots. The plotting of the graph of an equation provides a method for determining the roots of an equation in one variable regardless of its degree. For instance, it may be required to obtain the roots of the quadratic equation $x^2 - 5x + 6 = 0$. To do this we let $y = x^2 - 5x + 6$ and set down a table of corresponding values for x and y. Then we plot the graph as in the preceding examples.

EXAMPLE 5–5. Determine the roots of the equation $x^2 - 5x + 6 = 0$, graphically.

Solution:

x	-4	-3	-2	-1	0	$+1$	$+2$	$+2.5$	3	4	5
y	42	30	20	12	6	2	0	-0.25	0	2	6

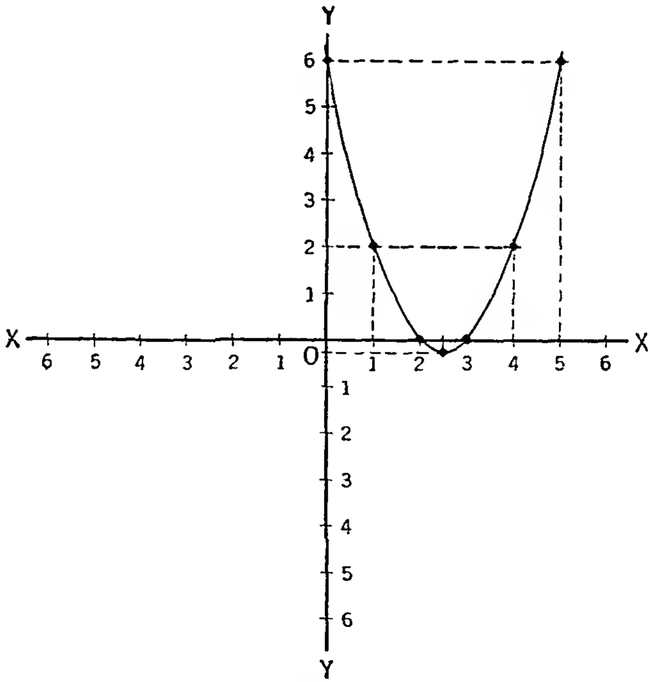


Fig. 5–8.

The roots of the equation $x^2 - 5x + 6 = 0$ are now found at the points where the curve intersects the x-axis. These points are seen to be at $x = +2$ and $x = +3$. Therefore, the roots of the equation $x^2 - 5x + 6 = 0$

are $x = +2$ and $x = +3$. This curve is a parabola since the equations contain the first and second powers of the variable.

The plotting of a quadratic function to determine its roots gives rise to three possible cases: The curve may intersect the x-axis in two points as in Fig. 5–8 and give two roots that are real. The curve may be tangent to the x-axis at one point, in which case the two roots are real and equal. Or the curve may not touch the x-axis, in which case the roots are imaginary. It is evident, therefore, that the maximum number of real roots for a second degree equation is two.

For equations of a higher order than the second, the plotting is done in exactly the same way as in Example 5–5. The maximum number of roots can be no greater than the highest power in the equation but it does not necessarily follow that there will be as many roots as indicated by the highest power. The number of roots can best be determined by a careful plotting.

EXERCISE 5–1

Plot the following points on graph paper and determine the distance of each from the origin:

1. $(3, 4)$; $(-8, +2)$; $(-6, -5)$; $(2, -5)$.
2. $(7, -5)$; $(8, 6)$; $(-7, -3)$; $(3, -9)$.
3. (a) Using the x-axis for Fahrenheit degrees and the y-axis for centigrade degrees, plot a graph showing the relation between the two scales.

$$0° \text{ centigrade} = 32° \text{ Fahrenheit.}$$
$$100° \text{ centigrade} = 212° \text{ Fahrenheit.}$$

(b) From the above graph determine the value of 72° Fahrenheit in degrees centigrade. Also determine the value of 70° centigrade in degrees Fahrenheit.

4. Plot the graph of the equation $4x - y = 6$.
5. Plot the graph of the equation $S = \frac{1}{2}gt^2$, using t as the independent variable and S as the dependent variable. Use $g = 32$.
6. Plot the graph of the equation $x_c = \frac{1}{2\pi fc}$ in which $c = 0.000025$ and f varies from 0 to 70. Plot f as abscissa and x_c as ordinate.
7. Determine graphically the roots of the equation $2x^2 - 3x - 2 = 0$.
8. Determine graphically the roots of the equation $2x^2 - 8x + 6 = 0$
9. Determine graphically the roots of the equation
$$6x^3 + 13x^2 - 14x + 3 = 0.$$
10. Determine graphically the roots of the equation
$$2x^3 - 21x^2 + 74x - 85 = 0.$$

5. Equation of the straight line. A function having the form $y = mx + b$, where m and b are constants, is one general form for the equation of a straight line and is termed a *linear equation.*

If we substitute any definite values for m and b and plot the graph of the resulting function, it will be found to be a straight line, and therefore we may say that the equation of a straight line may always be written in the form

$$y = mx + b.$$

The equation of Example 5–2 is an example of a straight line. This equation, $3x + 2y = 8$, can be put in the form $y = mx + b$ by solving for y; thus,

$$3x + 2y = 8$$

or

$$2y = -3x + 8$$

or

$$y = -\frac{3}{2}x + 4.$$

This last equation then is in the form $y = mx + b$, where $m = -\frac{3}{2}$ and $b = 4$.

In the general equation $y = mx + b$, if we let $m = 1$ and $b = 0$, then the resulting equation $y = x$ is represented by a line in a fixed position with reference to the x- and y-axes. The line passes through the origin and makes an angle of 45° with each axis. Thus the abscissa and ordinate of each point on the line will be equal.

Again, if $b = 0$ and m is given any value, such as 3, the resulting equation, $y = 3x$, is represented by a line equally definite in position, but the ordinates in this case are always three times the abscissas. The line still passes through the origin, but its position with reference to the axes is changed. We might substitute any other value for m, but so long as we make $b = 0$ the line will pass through the origin.

Fig. 5–9 shows the two lines $y = x$ and $y = 3x$ plotted.

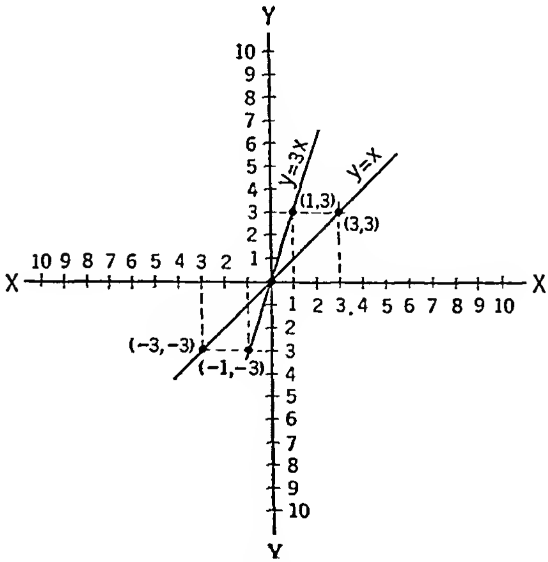


Fig. 5–9.

From the foregoing it is evident that any change in the value given to m changes the angle that the line makes with the x-axis. This value of m is called the *slope of the line* and may be defined as the ratio of the increase in the ordinate to the increase in the abscissa. In the equation $y = 3x$ the slope of the line is 3.

A line will have a *positive slope* if the angle from the positive x-axis to the line, measured in a counterclockwise direction, is an acute angle (less than 90°); and a *negative slope* if this angle is an obtuse angle (greater than 90°).

To determine the slope of a line, select any two points as far apart as convenient and divide the change in y by the change in x. If a different scale is used for the y-values than is used for the x-values, the angle of the line with the x-axis will not be the same as for like scales. But the *slope* of the line will still be the *change in y divided by the change in x.* The *slope* of a line is the *tangent of the angle made with the x-axis.*

Now suppose that some value other than zero is assigned to b. For example, in the general equation, $y = mx + b$, suppose $m = 3$ and $b = 2$. Then we have $y = 3x + 2$. This means that y is three times as great as x and in addition is increased by two more units. When the equation for this line is plotted, it will be noticed that it is two units higher along the y-axis than the equation $y = 3x$. If the equation $y = 3x - 2$ were to be plotted it would be found to be two units lower along the y-axis than the equation $y = 3x$. But it also would be found that each of these lines, $y = 3x + 2$ and $y = 3x - 2$, would be parallel to $y = 3x$.

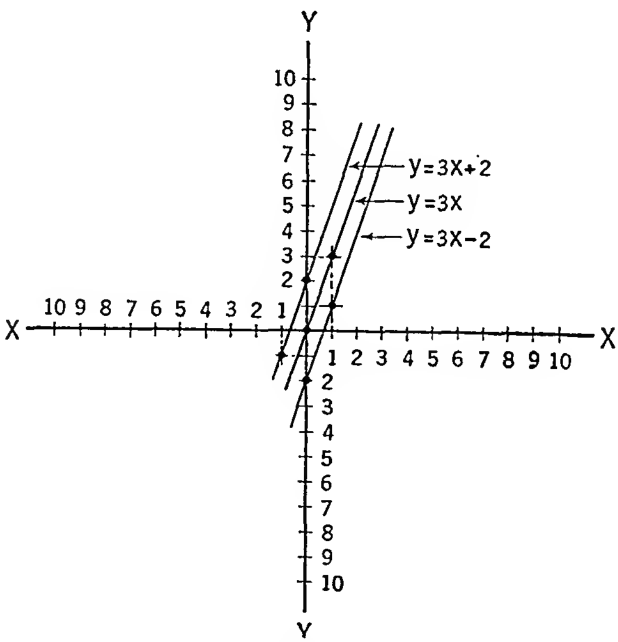


Fig. 5–10.

It follows, then, that the constant b represents the point at which the straight line intersects the y-axis, and therefore it is called the *y-intercept*, and the equation $y = mx + b$ is termed the *slope-point form* of the equation of a straight line because it is determined from the slope m of the line and one point on the line, which is b, the y-intercept. Fig. 5–10 shows the three equations, $y = 3x$, $y = 3x + 2$, and $y = 3x - 2$, plotted.

It may be that the point at which the line intersects the x-axis is desired. This point is the *x-intercept* and may be found by letting $y = 0$ in the original equation and solving for x. Thus in the equation $y = 3x + 2$, if $y = 0$ we have

$$3x + 2 = 0, \qquad 3x = -2, \qquad x = -\frac{2}{3}.$$

Therefore, the x-intercept or the point where the line crosses the x-axis is $-\frac{2}{3}$.

EXAMPLE 5–6. Write the equation of the line passing through the point (2, 4) and having a slope of +5.

Solution: Substituting the value of +5 for m in the general equation $y = mx + b$, there results

$$y = 5x + b.$$

Since the point (2, 4) must be on the line, we can substitute these values for x and y respectively. Thus,

$$4 = (5)(2) + b$$

or $$b = 4 - 10 = -6.$$

The required equation is $$y = 5x - 6.$$

EXAMPLE 5–7. Find the equation of the line passing through the points (5, 0) and (1, 2).

Solution: Substituting each set of values in the general equation $y = mx + b$, there results

$$0 = (m)(5) + b,$$
$$2 = (m)(1) + b,$$

or $$b = -5m$$

and $$m + b = 2.$$

Solving these two simultaneously for values of m and b, we get

$$m = -\frac{1}{2}, \qquad b = +\frac{5}{2}.$$

Thus, the equation of the line passing through the points (5, 0) and (1, 2) is $y = -\frac{1}{2}x + \frac{5}{2}$.

This may be written as $2y = -x + 5$.

EXERCISE 5-2

1. Construct a graph showing the relationship between inches and centimeters from the following table and determine the equation:

Inches	*Centimeters*
0	0.000
1	2.54
2	5.08
3	7.62
4	10.16
5	12.70
6	15.24

Find the equation of the line in each of the following conditions:

2. Passes through the point (1, 3) and has a slope of $\frac{1}{3}$.
3. Passes through the points (5, 5) and (3, 4).
4. Passes through the point $(-1, -2)$ and has a slope of $-1\frac{1}{2}$.
5. Passes through the points (0, 0) and $(2, -2)$.

6. Determination of equation from experimental data. Very often it happens that data obtained by experiment will be somewhat inaccurate owing to errors in reading instruments, faults in equipment, changes in conditions affecting the experiment, or a combination of several factors. As a result, the data gathered, while indicating a relationship, will vary to such an extent that a curve cannot be drawn through all the plotted points representing the readings taken in the experiment.

This condition occurs very frequently and presents a problem in determining where the curve actually should be. The methods already described for computing the equation of a straight line cannot be used except in cases where all points observed in the experiment fall on the line. However, there are methods by which the equation can be determined from the observed data.

Since a very important straight-line characteristic is the one in which the differences of the y values for equal difference of x are equal, we can determine quickly whether the relationship can be expressed at once in the form of a straight line, by tabulating a few values of y for values of x that are equally spaced. If the differences in these y values are the same, then the straight-line equation applies. If not, then some other method must be used.

EXAMPLE 5-8. Determine if the following values of frequency, f, and reactance, X, taken on an iron core coil, fit a straight line of the form

$$X = Mf + b.$$

f	X
30	60
40	75
50	90
60	105
70	120
80	135

Solution: An inspection of the readings shows that a difference of 10 in the f values makes a difference of 15 in the X values for any of the readings. Therefore, the observed readings fit a straight line. The equation, then, may be determined from any two sets of values.

Thus, substituting in $x = Mf + b$ the first two sets of values, we have

$$60 = (M)(30) + b,$$
$$75 = (M)(40) + b.$$

Solving simultaneously, $$M = \frac{3}{2},$$
$$b = 15.$$

Then the equation is $$x = \frac{3}{2}f + 15.$$

In cases where the straight-line equation does not fit all the data, and these cases are more numerous than otherwise, there are other methods that may be used to determine the equation or graph that best fits the data.

Method 1 is a *graphical method* and consists of drawing a line so that the points are divided about equally on each side of the line. This method may be used quite successfully in a great many cases. An example of this method is the graph of voltage and field current in Fig. 5–1.

Method 2 is called the *method of moments.*

EXAMPLE 5–9. Determine the equation for the following values of x and y:

x	20	30	40	50
y	48	66	80	98

Solution: Assuming a straight-line equation, we substitute these values in the general form $y = mx + b$ and we get the following:

$$20m + b = 48,$$
$$30m + b = 66,$$
$$40m + b = 80,$$
$$50m + b = 98.$$

Now multiply each equation by the coefficient of b in that equation and add the resulting equations. This addition gives

$$140m + 4b = 292. \quad (1)$$

Next multiply each equation by the coefficient of m in that equation and add the resulting equations:

$$\begin{aligned} 400m + 20b &= 960 \\ 900m + 30b &= 1{,}980 \\ 1{,}600m + 40b &= 3{,}200 \\ 2{,}500m + 50b &= 4{,}900 \\ \hline \text{adding:} \quad 5{,}400m + 140b &= 11{,}040. \quad (2) \end{aligned}$$

Solving equations (1) and (2) simultaneously,

$$m = 1.64 \text{ and } b = 15.6.$$

Then the equation is $\qquad y = 1.64x + 15.6.$

Now if we check back and compute values of y for the value of x given originally, we get

x	20	30	40	50
y	48.4	64.8	82.2	97.6

Method 3 is called the *method of averages.*

Example 5–10. Determine the equation for the values of x and y given in Example 5–9.

Solution: Substituting the values in the general form of the equation gives

$$20m + b = 48, \tag{1}$$
$$30m + b = 66, \tag{2}$$
$$40m + b = 80, \tag{3}$$
$$50m + b = 98. \tag{4}$$

Adding (1) and (2), we get

$$50m + 2b = 114. \tag{5}$$

Adding (3) and (4), we get

$$90m + 2b = 178. \tag{6}$$

Solving (5) and (6) simultaneously, we get

$$m = 1.6 \text{ and } b = 17.$$

Then the equation is $\qquad y = 1.6x + 17.$

If we were to add (1) and (3) and then (2) and (4), we would get

$$60m + 2b = 128,$$
$$80m + 2b = 164.$$

Solving these last two,

$$m = 1.8 \text{ and } b = 12.$$

The equation then would be

$$y = 1.8x + 12.$$

The method best suited for a particular problem can be determined by the appearance of the data when plotted. In general the method of moments will be the most accurate, though other methods may be shorter.

EXERCISE 5–3

1. Determine the equation of the form $E = mR + b$ for the following values of effort E and load R that were observed in a test on a crane:

R	10	20	30	40	50
E	6	10.8	16.1	20.8	26

2. The following readings were taken on a standard gas meter s and a meter T under test. Determine the equation of the form $T = ms + b$ that fits the data.

s	3,000	3,510	4,022	4,533
T	0	500	1,000	1,500

3. The pressure P in pounds of water at different depths D in feet is given in the following table. Draw the graph and determine the equation of the form $P = MD + b$.

D	6	10	20	30	40	50	60	70	80
P	2.6	4.33	8.66	12.99	17.32	21.65	25.99	30.32	34.65

4. The readings in the following table were taken to show the density d of liquid ammonia at various degrees of temperature t centigrade. Determine the equation of the form $d = mt + b$.

t	0	5	10	15
d	0.6364	0.6298	0.6230	0.6160

5. The speed of a shunt motor is determined from the equation

$$s = K\frac{V - Ia\,Ra}{\Phi}.$$

Since the only variables in this equation are s and I_a and all the other terms are constant, it could be put in the form

$$s = b - mI.$$

Determine the straight line of this form and plot the graph that fits the following reading of s and I:

I	0	3	5.5	8	10.5	12
s	1,750	1,740	1,730	1,720	1,710	1,700

6. The specific heat s of hot liquid ammonia is given in the following tables for various degrees Fahrenheit t. Determine the straight-line relationship of the form $s = mt + b$.

t	5	10	15	20	25
s	1.09	1.084	1.078	1.072	1.066

7. **Graphical solution of simultaneous equations.** If we were to take the two equations $x + y = 8$ and $2x - y = 7$ and plot the graph of each

one, we would find that the two lines would intersect in one point. As shown in Fig. 5-11, this point is (5, 3).

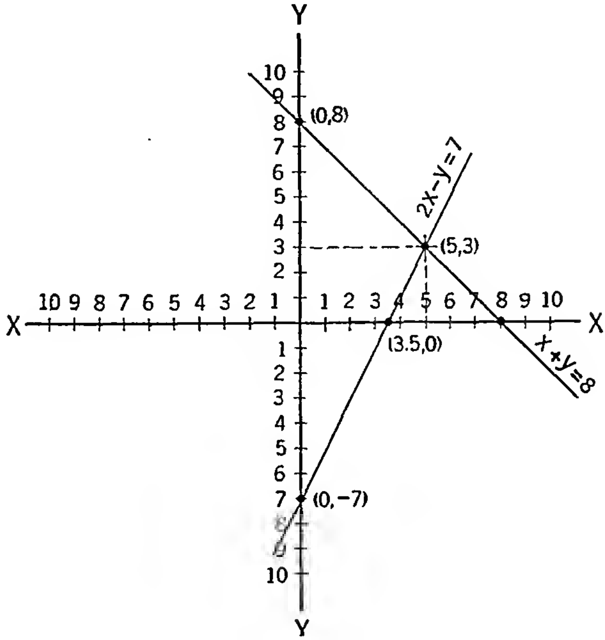


Fig. 5-11.

Since the point (5, 3) is on each line, it satisfies the equation for each line and thus represents the one set of values common to both lines. It is therefore a solution for these two equations simultaneously.

Any other two equations can be solved simultaneously in a similar way. If two simultaneous equations are given to be solved graphically, each equation is plotted and the points where the graphs intersect are determined. Such points will be the values for the simultaneous solution of the equations.

If the plotted graphs do not intersect, the equations are inconsistent and have no simultaneous values.

If the graphs of two equations, one of which is a higher order, fail to intersect but are tangent at some point, there is only one set of values for their simultaneous solution.

Example 5-11. Solve the following equations simultaneously:

$$x^2 + y^2 = 25;$$
$$x - y = 1.$$

Solution: As shown in Fig. 5-12, the points (4, 3) and (−3, −4) are the points where the straight line $x - y = 1$ intersects the circle $x^2 + y^2 = 25$. These points (4, 3) and (−3, −4) then are the values for the simultaneous solution of these equations.

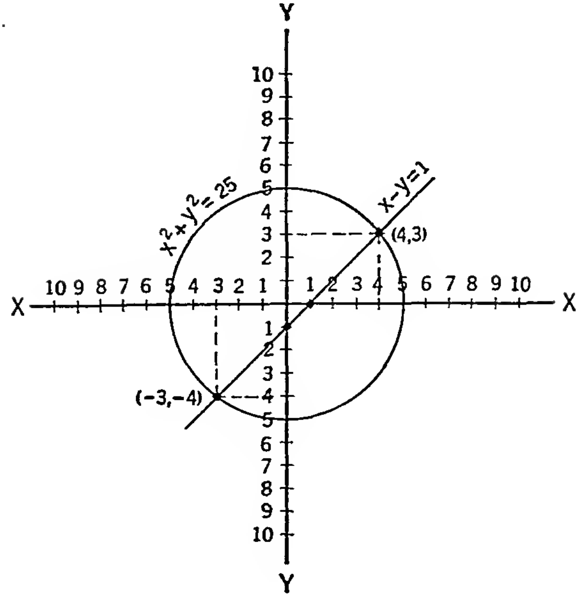


Fig. 5-12.

EXERCISE 5-4

Solve graphically the following sets of equations:

1. $4x + 3y = 12,$
 $x = y - 1.$
2. $x^2 + y^2 = 5,$
 $xy = 2.$
3. $x^2 + y^2 = 9,$
 $y = 3x - 4.$

4. Determine the value of K that will make the straight line $x + y = K$ tangent to the circle whose equation is $x^2 + y^2 = 36$. What values will make the line intersect the circle? What values will place the line outside the circle entirely?

Hint: Eliminate y from the two equations and use only the discriminant of the resulting quadratic, which should be set equal to zero in order to solve for K. See Section 24, Chapter 4, Part I, for a discussion of the discriminant of a quadratic equation.

REVIEW EXERCISE 5-5

Plot the following points on graph paper and determine the distance of each from the origin:

1. $(5, 7)$.
2. $(-9, +4)$.
3. $(-12, -8)$.
4. $(+14, -12)$.
5. $(-15, +6)$.
6. $(15, 12)$.
7. $(+25, -18)$.
8. $(-17, -22)$.
9. $(-18, +14)$.
10. $(+16, -13)$.

Plot the graph for each of the following equations:

11. $2x + 3y = 8$.
12. $5y - x = 12$.
13. $4x - 7y = 15$.
14. $x + 4y = -6$.
15. $-3x + 5y = -10$.
16. $2x - 7y = -9$.
17. $x^2 + y^2 = 36$.
18. $x^2 + 2x + y^2 + 2y = 0$.
19. $x^2 - y^2 = 16$.
20. $y^2 = 6x$.
21. $x^2 = 4y$.
22. $4y^2 = 16 - x^2$.
23. $x^2 = 16 + 4y^2$.
24. $x^2 - 2x = y^2 - 1$.
25. $9y^2 - 36 = -4(x - 3)^2$.

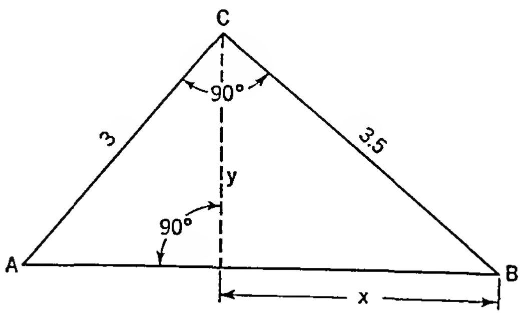
Determine graphically the roots of the following equations:

26. $y = 2x^2 - x - 6$.
27. $y = 3x^2 - 10x - 8$.
28. $y = 2x^2 + x - 15$.
29. $y = 8x^3 - 44x^2 - 36x + 20$.
30. $y = 6x^3 + 27x^2 - 18x - 120$.

Solve the following simultaneous equations by graphical methods:

31. $x + 3y = 4$, $2x - y = 5$.
32. $2y - 3x = 12$, $4y + x = 10$.
33. $3y + 2x = 8$, $4x - 5y = 10$.
34. $5x - 4y = 12$, $3y + 6x = 11$.
35. $y^2 = 49 - x^2$, $y = x - 1$.
36. $x^2 - 40 = y^2$, $2x - y = 12$.
37. $2y = x - 2$, $4y^2 = 16 - x^2$.
38. $x^2 + y^2 - 4x - 8y + 20 = 0$, $x + y = 3$.
39. $(x - y)^2 = 4$, $x^2 + y^2 = 4$.
40. $x^2 + y^2 = 32$, $xy = 16$.

41. Find x and y in the right triangle ABC.



Note that the logarithm of 3.162, a number between 1 and 10, is 0.500, a number between 0 and 1. Also, the logarithm of 2.154, a number between 1 and 10, is 0.3333, a number between 0 and 1.

Any number may be considered as a power of 10 just as we have considered 0.5000 and 0.3333 as powers of 10. Thus, 10 raised to the 1.6990 power gives the number 50. In logarithmic notation this is

$$\log_{10} 50 = 1.6990.$$

The logarithm is composed of two parts: a whole number and a decimal. The whole number part is called the *characteristic* and the decimal part is called the *mantissa*. The mantissa, by common agreement, is always considered as positive in value while the characteristic may be either positive or negative in value. Negative characteristics will be treated later. In logarithms to the base 10, the decimal part or mantissa is always the same for a given sequence of numbers. It does not change when the decimal point is moved. Only the characteristic changes for a change in the location of the decimal point.

Examples:

$$\begin{aligned} \log 5.0 &= 0.6990, \\ \log 50 &= 1.6990, \\ \log 500 &= 2.6990. \end{aligned}$$

The following rule for the determination of the characteristic of a logarithm is a direct development from the conversion table for powers of 10 into logs:

For a number greater than 1, the characteristic of its logarithm is positive, and, in numerical value, is one less than the number of digits at the left of the decimal point in the number. Thus,

$$\begin{aligned} \log_{10} 3.915 &= 0.59273, \\ \log_{10} 39.15 &= 1.59273, \\ \log_{10} 391.5 &= 2.59273, \\ \log_{10} 3915 &= 3.59273. \end{aligned}$$

Note that the characteristic in each case is one less numerically than the number of digits that appear to the left of the decimal point in the number.

Logarithms to the base 10 have been computed for all numbers and placed in convenient tables. Tables to four, five, six, seven, and sometimes more places have been computed. Four- and five-place tables are used for most calculations. Part of a typical four-place log table is shown on page 211. Only the mantissa, or decimal part, of the logarithm is given in the table. The characteristic must be supplied.

The column at the left, headed by the letter N, gives the first two figures in the number. The third figure is found at the top of one of the other columns headed 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2579	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201

EXAMPLE 6–1. Find the logarithm of 184.

Solution: Find 18 under the N column. Read across to the right opposite 18 until the column under 4 is reached. Here we find the number 2648. Then the decimal part or mantissa of the log of 184 is 0.2648. The characteristic of this log must be 2, since 184 is between 100 and 1,000 and the log of 100 is 2, whereas the log of 1,000 is 3. So

$$\log 184 = 2.2648.$$

Note that the mantissa for the sequence of numbers 184 is always 0.2648 even though the characteristic may change. Thus,

$$\begin{aligned}\log 1.84 &= 0.2648,\\ \log 18.4 &= 1.2648,\\ \log 184 &= 2.2648,\\ \log 1840 &= 3.2648.\end{aligned}$$

From this it also may be seen that the mantissa of a number is independent of the position of the decimal point in the number, but that the characteristic depends entirely upon the position of the decimal point.

In a set of five-place log tables, there will be found in the first vertical column a list of numbers from 0 to 1,000. The mantissa of the log of any one of these numbers may be found opposite the number in the second vertical column. Thus, the mantissa for the number 183 is 0.26245; for 195 is 0.29003; and for 107 is 0.02938. As before, each vertical column is headed by a number (from 0 to 9) and these columns give an extra figure in the number whose log we desire to find. Thus, we may want the log of a number with four figures such as 1,725. Since the column at the left gives numbers only to 1,000, it is clear that we can find only the number 172 here. But the column headed by the figure 5 will give us the last figure in the number, 1,725.

Since the first two figures of the mantissa will be the same for several lines, they will appear in the table only in the first line and the first column. This is done to avoid repetition. If an asterisk appears in the last line for which a certain two first figures apply, it indicates that the next two first figures are to be used beginning at that point. Thus, the mantissa of the log of 6,456 is 0.80996, the first two figures, 80, being chosen from the first

column and from the line opposite 631. In the column for 6,457 an asterisk appears with the three numbers 003. This means that the first two figures for the mantissa will be 81 instead of 80 and the complete mantissa will be 81003.

The symbols $\bar{5}$ and $\dot{5}$ are also found at some places in the table and they are used to indicate how the last figure 5 has been derived. For instance, the log 8.8307$\bar{5}$ is given more fully as 8.8307495, while the log 9.4082$\dot{5}$ is given more fully as 9.4082539.

EXAMPLE 6–2. Find the log of 6,748.

Solution: Find the number 674 in the column at the left. Proceed to the right horizontally opposite 674 until the column headed by the figure 8 is reached. Then read the mantissa of the logarithm. It will be 0.82918. Since the original number is between 1,000 and 10,000, the characteristic is 3, and the complete logarithm is 3.82918. Therefore, $\log_{10} 6748 = 3.82918$.

EXERCISE 6–1

Find the logs of the following:

1. 156.8.
2. 1.427.
3. 84.56.
4. 3.260.
5. 604.
6. 46.25.
7. 2,035.
8. 3.812.
9. 136.5.
10. 27.03.
11. 600,000.
12. 6,500,000,000.
13. 24.850.
14. 17,000,000.
15. 3,250,000.
16. 3.142.
17. 1.732.
18. 141.4.
19. 7,945.
20. 126,000,000.

4. Negative characteristics. Thus far we have discussed only the logarithms of numbers larger than 1. But numbers smaller than 1 are used and we must find a way of handling them by logs.

Going back to our laws of exponents again, we know that

$$10^{-1} = \frac{1}{10} = 0.1,$$

$$10^{-2} = \frac{1}{100} = 0.01,$$

$$10^{-3} = \frac{1}{1,000} = 0.001,$$

$$10^{-4} = \frac{1}{10,000} = 0.0001.$$

Then, in logarithmic notation:

$$\begin{aligned}\log 0.1 &= -\ 1.0000 = {}^{-}1.0000,\\ \log 0.01 &= -\ 2.0000 = {}^{-}2.0000,\\ \log 0.001 &= -\ 3.0000 = {}^{-}3.0000,\\ \log 0.0001 &= -\ 4.0000 = {}^{-}4.0000.\end{aligned}$$

Now we can see that numbers smaller than 1 will have negative characteristics for their logarithms. Since only the characteristic may be negative, the minus sign to denote a negative characteristic is usually placed above the characteristic instead of in front of it.

Since the mantissa is always positive and the characteristics are negative for these numbers that are smaller than 1, it is difficult to use the logarithm in this form. Therefore, 10, or a multiple of 10, is added to the characteristic and subtracted at the right of the mantissa. Thus, $\bar{1}$ becomes $9 - 10$; $\bar{5}$ becomes $5 - 10$, $\overline{12}$ becomes $8 - 20$, and so on. This does not change the value of the logarithm because it has the effect of adding zero since $10 - 10$ equals zero. It does, however, put the logarithm in the form $K - 10$, and this form is more convenient to use because we do not have a mixture of positive and negative numbers to add. Examples are:

$$\log 0.135 = \bar{1}.13033 = 9.13033 - 10$$
$$\log 0.0162 = \bar{2}.20952 = 8.20952 - 10$$
$$\log 0.00208 = \bar{3}.31806 = 7.31806 - 10$$

The following rule provides an easy way of remembering the negative characteristic:

For numbers smaller than 1, the characteristic will be negative and numerically one greater than the number of zeros between the decimal point and the first significant figure.

EXAMPLE 6–3. Find the log of 0.0184.

Solution: The mantissa will be the same as in Example 6–1, since the sequence of numbers is the same, but the characteristic will be $\bar{2}$. Thus,

$$\log 0.0184 = \bar{2}.2648 \quad \text{or} \quad 8.2648 - 10.$$

A simple way by which to determine the characteristic of a logarithm, regardless of whether it be positive or negative, is to write the number whose log is desired in scientific notation form where the number is expressed as a number between 1 and 10 multiplied by the proper power of 10. The characteristic of its logarithm will then always be that power of 10. The following table gives illustrative cases:

Number	*Scientific notation*	*Characteristic of logarithm*
32765	3.27×10^4	4
158	1.58×10^2	2
15	1.5×10^1	1
2	2×10^0	0
0.00387	3.87×10^{-3}	$\bar{3}$ or $7 - 10$
0.00000452	4.52×10^{-6}	$\bar{6}$ or $4 - 10$
0.2	2×10^{-1}	$\bar{1}$ or $9 - 10$

5. Different ways of writing the characteristic. Thus far we have written the characteristic of a logarithm in the form of a whole positive

number at the left of the mantissa or in the form $K - 10$. Other forms may be encountered, however, and it is well to become familiar with all these forms. Thus,

$$\begin{aligned} \log 625 &= 2.79588 \\ &= 12.79588 - 10 \\ &= 22.79588 - 20 \\ &= 32.79588 - 30 \\ &= 0.79588 + 2. \end{aligned} \qquad \text{by adding and subtracting multiples of 10}$$

$$\begin{aligned} \log .00625 &= -3.79588 \\ &= 7.79588 - 10 \\ &= 17.79588 - 20 \\ &= 27.79588 - 30 \\ &= 0.79588 - 3. \end{aligned} \qquad \text{by adding and subtracting multiples of 10}$$

The operation in this last step can actually be carried out leaving a complete negative value of -2.20412 for the log of .00625.

To multiply .00625 by 5 it is necessary only to add their logs and look up the antilog in the tables. Any one of the forms above may be used. The ordinary method would be as follows:

$$\begin{aligned} &\log .00625 = 7.79588 - 10 \\ &\log \qquad 5 = \underline{0.69897 \qquad\;\;} \\ &\text{combining} \quad 8.49485 - 10 \\ &\text{antilog} \quad = \;\; .03125. \end{aligned}$$

The solution by means of the whole negative log would be:

$$\begin{aligned} &\log \qquad 5 &&= +10.69897 - 10 \\ &\log .00625 = 0.79588 - 3 &&= \underline{-\;\;2.20412 \qquad\;\;} \\ &\qquad\qquad \text{combining} && \qquad 8.49485 - 10 \\ &\qquad\qquad \text{antilog} = && \qquad .03125. \end{aligned}$$

Note that in the second solution, log 5 is written with the characteristic in the form $10 - 10$ in order to subtract from it the log of .00625 written in the complete negative form.

EXERCISE 6-2

Express in scientific notation and find the logs of the following:

1. 0.278.
2. 0.0542.
3. 0.624.
4. 0.00087.
5. 0.002047.
6. 0.0000007248.
7. 0.09126.
8. 0.5659.
9. 0.0005627.
10. 1.59×10^{-19}.
11. 9.0×10^{-28}.
12. 0.003142.
13. 0.07466.
14. 0.2828.
15. 0.04315.
16. 0.001231.
17. 0.1973.
18. 0.0003821.
19. 0.00002780.
20. 3.587×10^{-6}.

6. Interpolation. Occasionally, it may be necessary to get the logarithm of a number some of whose figures cannot be found in the table. For example, it may be required to find the log of 3,765.4. Now from our five-place tables we can get the log of 3,765 but we cannot get the rest directly. The process whereby such logarithms are found is called *interpolation,* and extends the use of the tables by one place. The method can be best treated by examples.

EXAMPLE 6–4. Find the log of 3,765.4.

Solution: First find the mantissas of 3,765 and 3,766. These will be respectively 0.57576 and 0.57588. The difference between the two mantissas is 0.00012. The difference between the numbers 3,765 and 3,766 is 1, and the difference between 3,765 and 3,765.4 is 0.4. Now, it is plain that the number 3,765.4, whose log we want, is 0.4 of the way between 3,765 and 3,766. Therefore, the log of 3,765.4, for all practical purposes, will be 0.4 of the way between the logs of 3,765 and 3,766. In other words, the difference between the mantissas of 3,765.4 and 3,765 will be 0.4 of the difference between the mantissas of 3,766 and 3,765, or 0.4 of 0.00012, which is 0.000048. Therefore, we must add this difference to the mantissa of 3,765 to get the mantissa of 3,765.4, the result being 0.575808, or 0.57581 in a five-place table.

$$\begin{aligned}
\log 3{,}766 &= 3.57588 \\
\log 3{,}765 &= \underline{3.57576} \\
\text{difference} &= 0.00012 \\
0.4 \times 0.00012 &= 0.000048 \\
\log 3{,}765.4 &= \log 3765 + (0.4)(0.00012) \\
\log 3{,}765.4 &= \quad 3.57576 \\
&\quad \underline{+\ 0.000048} \\
&\quad 3.575808 \text{ or} \\
&\quad 3.57581 \text{ to five places.}
\end{aligned}$$

EXAMPLE 6–5. Find the log of 782.463.

Solution:

$$\begin{aligned}
\log 782.5 &= 2.89348 \\
\log 782.4 &= \underline{2.89343} \\
\text{difference} &= 0.000050
\end{aligned}$$

Since 782.463 is 0.63 of the way between 782.4 and 782.5, then the log of 782.463 also will be 0.63 of the way between the logs of 782.4 and 782.5. So (0.63)(0.00005) = 0.0000315, or 0.00003 to five places. Adding this to the log of 782.4,

$$\begin{aligned}
\log 782.4 &= 2.89343 \\
(0.63)(0.00005) &= \underline{0.00003} \\
\log 782.463 &= 2.89346.
\end{aligned}$$

Interpolation also can be performed by using the tables of proportional parts that are included with the logarithmic tables. These tables show the parts of the tabular differences that correspond to the digits from 1 to 9

in the fifth place of the logarithm. In effect, the tabular difference is changed into tenths for the fifth place in the log. In the sample at the left the number at the top is the tabular difference, in this case, 12. The column at the left indicates tenths and the column at the right indicates the tenths of the tabular difference, 12. Thus, to find three tenths of 12, look down the left-hand column until you find the figure 3, then opposite this 3 you will find 3.6 in the right-hand column, which is three tenths of 12. Since interpolating is done to get the log accurately to five places, the value 3.6 is rounded off to 4.

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

In Example 6–4, the tabular difference was 12 and it was required to obtain four tenths of 12. In the table of proportional parts, four tenths of 12 is 4.8, which is rounded off to 5. Adding 5 to the fifth place of the log 3.57576 gives 3.57581.

EXERCISE 6–3

Find the logs of the following:

1. 0.34825.	6. 1,048.6.	11. 39,862.
2. 6,837.3.	7. 0.063515.	12. 7.7654.
3. 983.51.	8. 328.75.	13. 660.77.
4. 0.0037468.	9. 7.2843.	14. 0.031694.
5. 100,060.	10. 41.567.	15. 97.798.

7. To find the number corresponding to a given logarithm. Thus far we have been finding the logarithm of a given number, but just as often it is necessary to find the number corresponding to a given logarithm, which is the reverse process. This will be illustrated by examples.

Example 6–6. Find the number whose logarithm is 1.91355.

Solution: Locate 91,355 in the table of mantissas. The number in the column N opposite 91,355 is 819. Also, the mantissa, 91,355, is in the column headed by the figure 5. Then the complete number, of which 91,355 is the mantissa, is 8,195. Now, since the characteristic of the log is $+1$, the number we desire must be between 10 and 100. Therefore, the number of which 1.91355 is the logarithm is 81.95. This number, 81.95, is called the *antilogarithm* of 1.91355.

Example 6–7. Find the antilogarithm of 2.61411.

Solution: We cannot find this exact mantissa in the table, but we can find 61,405 and 61,416, whose antilogs are 4,112 and 4,113 respectively. The difference between these mantissas is 11, and the difference between the smaller one, 61,405, and our mantissa is 6. Therefore, the antilog we desire is $\frac{6}{11}$ of the way between 4,112 and 4,113. Since $\frac{6}{11}$ is 0.5 approximately, our antilog will be 411.25. The decimal point is located between

the 1 and 2 because the characteristic is 2, which places the number between 100 and 1,000.

$$\left.\begin{array}{r} \text{mantissa of } \log 4{,}113 = 61{,}416 \\ \left.\begin{array}{r}\text{mantissa of } \log x = 61{,}411 \\ \text{mantissa of } \log 4{,}112 = 61{,}405\end{array}\right\}\text{diff} = 6 \end{array}\right\}\text{diff} = 11$$

Therefore, the antilog of 61,411 is $\frac{6}{11}$ of the way between 4,112 and 4,113, or equals 411.25.

The interpolating can be performed by using the tables of proportional parts in the reverse order to that used in finding the logarithm. Thus, in the column for proportional parts of 11, which is the total tabular difference between the upper value of 61,416 and the lower value of 61,405, the right-hand column will show 5.5 as the value nearest to 6, the tabular difference between the given value of 61,411 and the lower value of 61,405. Opposite 5.5 in the left-hand column will be found 5, which will be the fifth figure of the antilog. Therefore, the complete antilog is 411.25.

EXERCISE 6-4

Find the antilogs of the following:

1. 1.04454.	6. $\overline{2}$.99958.	11. 2.67110.	16. $\overline{3}$.26615.
2. 2.72074.	7. 9.60221−10.	12. 0.21105.	17. 2.37757.
3. 8.30060−10.	8. 6.65386−10.	13. 8.43053−10.	18. 0.77130.
4. 3.89230.	9. 0.25438.	14. 3.56044.	19. 1.95020.
5. 0.90367.	10. 1.00060.	15. 1.08070.	20. $\overline{1}$.84207.

8. Properties of logarithms. Since a logarithm is an exponent, the rules relating to the use of logarithms are similar to the laws of exponents. In our study of Chapter 3 we found the laws of exponents to be the following:

$$\begin{aligned}
(x^m)(x^n) &= x^{m+n}, \\
\frac{x^m}{x^n} &= x^{m-n}, \\
(x^m)^n &= x^{mn}, \\
(xyz)^m &= x^m y^m z^m, \\
\left(\frac{x}{y}\right)^m &= \frac{x^m}{y^m}, \\
\sqrt[n]{x^m} &= x^{m/n}, \\
x^{-m} &= \frac{1}{x^m}, \\
x^0 &= 1.
\end{aligned}$$

These laws of exponents can be applied directly to logarithms, and the following basic rules can be derived:

1. The logarithm of a product is equal to the sum of the logarithms of the separate factors.

Let $\qquad A = x^m \quad \text{or} \quad \log_x A = m$
and $\qquad B = x^n \quad \text{or} \quad \log_x B = n.$
Then $\qquad AB = (x^m)(x^n) = x^{m+n}.$

In logarithmic form, this is

$$\log_x AB = m + n.$$

But $\qquad m + n = \log_x A + \log_x B.$
Therefore, $\qquad \log_x AB = \log_x A + \log_x B.$

In a similar way, the rule can be shown true for any number of factors.

2. The logarithm of a quotient is equal to the difference between the logarithm of the numerator and the logarithm of the denominator.

If $\qquad A = x^m \quad \text{or} \quad \log_x A = m$
and $\qquad B = x^n \quad \text{or} \quad \log_x B = n,$
then $\qquad \frac{A}{B} = \frac{x^m}{x^n} = x^{m-n}.$

In logarithmic form, this is

$$\log_x \frac{A}{B} = m - n.$$

But $\qquad m - n = \log_x A - \log_x B.$

Therefore, $\qquad \log_x \frac{A}{B} = \log_x A - \log_x B.$

3. The logarithm of a number that is raised to a power is the product of the exponent and the logarithm of the number.

If $\qquad A = x^m \quad \text{or} \quad \log_x A = m,$
then $\qquad (A)^n = (x^m)^n = x^{mn}.$

In logarithmic form, this is

$$\log_x A^n = mn.$$

But $\qquad m = \log_x A.$
Therefore, $\qquad \log_x A^n = n \log_x A.$

4. The logarithm of a root of a number is equal to the log of the number divided by the indicated root.

If $\qquad A = x^m \quad \text{or} \quad \log_x A = m,$
then $\qquad \sqrt[n]{A} = A^{1/n} = (x^m)^{1/n} = x^{m/n}.$

In logarithmic form, this is

$$\log_x \sqrt[n]{A} = \frac{m}{n}.$$

But $\qquad m = \log_x A.$

Therefore, $\qquad \log_x \sqrt[n]{A} = \frac{\log_x A}{n} = \frac{1}{n} \log_x A.$

9. Multiplication by the use of logarithms. From Rule 1, we find that the logarithm of a product is equal to the sum of the logarithms of the separate factors.

EXAMPLE 6–8. Evaluate $N = (15.872)(0.21486)(6.4713)$.

Solution: From the tables,

$$\begin{aligned}
\log 15.872 &= 1.20063 \\
\log 0.21486 &= 9.33216 - 10 \\
\log 6.4713 &= \underline{0.81099} \\
\log N &= 11.34378 - 10
\end{aligned}$$

or $\log N = 1.34378.$

Then $N = 22.069.$

If the problem contains negative quantities, they should be treated as if they were positive, because a negative number does not have a logarithm. The proper sign of the answer can be determined by inspection after the problem is completed.

EXAMPLE 6–9. Evaluate $N = (-1.743)(29.27)$.

Solution: From the tables,

$$\begin{aligned}
\log 1.743 &= 0.24130 \\
\log 29.27 &= \underline{1.46642} \\
\log N &= 1.71772 \\
N &= -52.206.
\end{aligned}$$

The answer is given a negative sign since the problem calls for multiplying a positive number by a negative number.

10. Division by the use of logarithms. From Rule 2, we find that the logarithm of a quotient is equal to the difference between the logarithm of the numerator and the logarithm of the denominator.

EXAMPLE 6–10. Evaluate $N = \dfrac{1{,}369.7}{17.92}$.

Solution: From the tables,

$$\begin{aligned}
\log 1{,}369.7 &= 3.13662 \\
\log 17.92 &= \underline{1.25334}
\end{aligned}$$

Subtracting, $\log N = 1.88328.$

$N = 76.433.$

When it becomes necessary to divide some number by a larger number, we are confronted with the problem of subtracting a larger logarithm from a smaller one. This is accomplished by putting the characteristic of the logarithm of the numerator in the form of $K - 10$.

EXAMPLE 6–11. Evaluate $N = \dfrac{152}{6{,}725}$.

Solution:

$$\begin{aligned}
\log 152 &= 2.18184 = 12.18184 - 10 \\
\log 6{,}725 &= 3.82769 = \underline{3.82769}
\end{aligned}$$

Subtracting $\log N = 8.35415 - 10.$

Then $N = 0.022602.$

11. Multiplication and division by the use of logarithms. When multiplication and division are combined in the same problem, Rule 1 and

Rule 2 both apply. The logarithm of the numerator and the logarithm of the denominator are determined separately by Rule 1. Then Rule 2 is applied to divide the numerator by the denominator.

EXAMPLE 6-12. Evaluate the following:

$$N = \frac{(5534)(0.02374)}{(8.963)(32.46)}.$$

Solution:

Numerator		*Denominator*	
log 5,534 =	3.74304	log 8.963 =	0.95245
log 0.02374 =	8.37548 − 10	log 32.46 =	1.51135
		log den. =	2.46380

$$\begin{aligned} \log \text{num.} &= 12.11852 - 10 \\ \log \text{den.} &= 2.46380 \\ \text{Subtracting} \quad \log N &= 9.65472 - 10. \\ N &= 0.45157. \end{aligned}$$

EXERCISE 6-5

Perform the following by logarithms:

1. $(14.37)(0.750)$.
2. $(6.258)(9{,}467.2)(4{,}381.5)$.
3. $(0.00854)(724.83)(36.285)$.
4. $(0.008623)(0.000427)(0.0000326)$.
5. $(6.72)(3{,}215.6)(148.6)(0.01045)$.
6. $\dfrac{938.26}{10.482}$.
7. $\dfrac{34.6}{5{,}287.5}$.
8. $\dfrac{0.93478}{7{,}283}$.
9. $\dfrac{75.48}{0.261}$.
10. $\dfrac{1.782}{36.939}$.
11. $\dfrac{(4.8275)(\pi)(0.32087)}{(5.7183)(0.004728)(981)}$.
12. $\dfrac{(75)(12.16)(8.375)(2.54)}{(2.7183)(0.43429)(91)(1563)}$.
13. $\dfrac{(483.7)(0.008721)(2.5 \times 10^{-6})}{(6.5 \times 10^{-8})(9.472)(0.6385)}$.
14. $\dfrac{(0.7854)(\pi)(1.732)}{(1.414)(2.59 \times 10^{10})}$.
15. $\dfrac{(0.636)(0.707)(0.009725)(4.38 \times 10^{8})}{(1.005)(4{,}167.8)(832.15)}$.

12. Raising to a power by the use of logarithms. From Rule 3, we know that the logarithm of a number that is raised to a power is the product of the exponent and the logarithm of the number. Therefore, to raise a number to a power, it is necessary only to get the logarithm of the number and multiply it by the desired power to get the logarithm of the result.

EXAMPLE 6–13. Raise 2 to the 12th power.

Solution:

$$\begin{aligned} \log 2^{12} &= 12 \log 2 \\ \log 2 &= 0.30103 \\ &\quad\ \ \underline{12} \\ &\quad\ \ \ 60206 \\ &\quad\ \ \underline{3\ 0103} \\ \log 2^{12} &= 3.61236 \end{aligned}$$

Then $$2^{12} = 4{,}096.$$

The problem becomes more complicated when a decimal number is to be raised to a decimal power. When the logarithm is multiplied by the power, the result comes out with a characteristic that is not in the form $K - 10$. To put it in this form a value is chosen that will make the negative part equal to -10, or some multiple of -10, and this value is added both positively and negatively to the logarithm.

EXAMPLE 6–14. Evaluate: $0.25^{0.41}$.

Solution:

$$\log 0.25 = 9.39794 - 10.$$

$$\log .25^{.41} = .41 \log .25,$$

Therefore, $$0.41(9.39794 - 10) = 3.85316 - 4.1.$$

Since -5.9 added to -4.1 gives -10, 5.9 is added positively and negatively to the log:

$$\begin{array}{rr} 3.85316 & -\ 4.1 \\ +5.9 & -\ 5.9 \\ \hline 9.75316 & -\ 10 \end{array}$$

Therefore, $$0.25^{0.41} = 0.56645.$$

EXAMPLE 6–15. Evaluate $\left(\frac{7.16}{95.8}\right)^{2/3}$

Solution:

$$\begin{array}{rrr} \log 7.16 = & 10.85491 & -\ 10 \\ \log 95.8 = & 1.98137 & \\ \hline & 8.87354 & -\ 10 \\ \text{multiplying by 2:} & 2 & 2 \\ \hline & 17.74708 & -\ 20 \\ \text{changing characteristic:} & 27.74708 & -\ 30 \\ \text{dividing by 3:} & 3\ |\underline{27.74708} & \underline{-\ 30} \\ & 9.24903 & -\ 10 \\ \text{antilog} = & 0.17743 & \end{array}$$

$$\text{Therefore } \left(\frac{7.16}{95.8}\right)^{2/3} = 0.17743.$$

EXAMPLE 6–16. Evaluate $.832^{-.52}$.

Solution:

$$.832^{-.52} = \frac{1}{.832^{+.52}}$$

$$\begin{array}{rl}
\log .832 = & 9.92012 - 10 \\
\text{multiplying by } .52\text{:} & \underline{.52 \quad .52} \\
 & 1984024 \\
 & \underline{4960060} \\
 & 5.1584624 \; -5.2 \\
\text{adding and subtracting } 4.8\text{:} & \underline{4.8 \qquad\quad -4.8} \\
\log .832^{+.52} = & 9.95846 \quad -10 \\
\\
\log 1 = & 10.00000 - 10 \\
\log .832^{+.52} = & \underline{9.95846 - 10} \\
\text{subtracting:} & 0.04154 \\
\text{antilog} = & 1.1004 \\
\text{Then } .832^{-.52} = & 1.1004.
\end{array}$$

13. Extracting a root by logarithms. From Rule 4, we find that the logarithm of a root of a number is equal to the logarithm of the number divided by the indicated root.

EXAMPLE 6–17. Evaluate $\sqrt[5]{76{,}285}$.

Solution:

$$\log 76{,}285 = 4.88244,$$

$$\frac{1}{5}\log 76{,}285 = 0.97649.$$

Therefore,

$$\sqrt[5]{76{,}285} = 9.473.$$

EXAMPLE 6–18. Evaluate $\sqrt[3]{0.8346^5}$.

Solution:

$$\begin{array}{rl}
\log 0.8346 = & 9.92148 - 10 \\
\text{multiplying by } 5\text{:} & \underline{5 \quad\; 5} \\
 & 49.60740 - 50 \\
\text{changing characteristic:} & 29.60740 - 30 \\
\text{dividing by } 3\text{:} & 3\,\underline{)29.60740 - 30} \\
 & 9.86913 - 10 \\
\text{antilog} = & 0.7398 \\
\text{Therefore } \sqrt[3]{0.8346^5} = & 0.7398.
\end{array}$$

EXERCISE 6–6

Perform the indicated operations in the following problems:

1. 2^{30}.
2. 17^7.
3. 0.0045^6.
4. 1.2843^3.
5. 728.56^4.
6. 0.00059^4.
7. 0.007128^2.
8. $0.515^{0.64}$.
9. $21.82^{0.62}$.
10. $0.039256^{0.33}$.
11. $\sqrt[3]{3.14159}$.
12. $\sqrt[30]{2.0000}$.

13. $\sqrt{17{,}584.305}$. 16. $\sqrt{0.0000042851}$. 19. $\sqrt[5]{785{,}632}$.
14. $\sqrt[4]{0.0215876}$. 17. $\sqrt{0.837500}$. 20. $\sqrt[3]{0.0006834}$.
15. $\sqrt{0.00005360}$. 18. $\sqrt[3]{76.1824}$.

14. Combination problems. In the majority of problems, there will be combinations of multiplication, division, raising to powers, and extracting of roots, but these combination problems present no additional difficulties. The multiplications of numerator and denominator are carried out separately and finally the total numerator divided by the total denominator is exactly the same as in a cancellation problem in arithmetic except that the work is performed by logarithms.

EXAMPLE 6–19. Evaluate the following problem:

$$N = \frac{(5{,}534)(0.02374)}{(\sqrt[3]{1{,}785})(3.246^3)(8.963)}.$$

Solution: *Numerator*

$$\begin{aligned} \log 5{,}534 &= 3.74304 \\ \log 0.02374 &= \underline{8.37548 - 10} \\ \log \text{numerator} &= 12.11852 - 10 \\ \log \text{denominator} &= \underline{3.57038} \\ \log N &= 8.54814 - 10 \\ N &= 0.03533. \end{aligned}$$

Denominator

$$\begin{aligned} \frac{1}{3}\log 1785 &= 1.08388 \\ (3)\ \log 3.246 &= 1.53405 \\ \log 8.963 &= \underline{0.95245} \\ \log \text{denominator} &= 3.57038. \end{aligned}$$

15. Cologarithms. In division problems, the cologarithm of the divisor may be used. Since the cologarithm of a number is the logarithm of its reciprocal, adding the cologarithm of the number is the same as subtracting its logarithm. Therefore, the process of division can be performed by addition in the same manner as multiplication. The cologarithm is obtained by subtracting the logarithm from 10.00000 − 10. Thus, the cologarithm of 1,785 is found as follows:

$$\begin{aligned} & 10.00000 - 10 \\ \log 1{,}785 &= \underline{3.25164} \\ \operatorname{colog} 1{,}785 &= 6.74836 - 10. \end{aligned}$$

By the use of cologarithms, the solution of Example 6–19 might be carried out thus:

$$\begin{aligned} \log 5{,}534 &= & & 3.74304 \\ \log 0.02374 &= & & 8.37548 - 10 \\ \log \sqrt[3]{1{,}785} &= 1.08388 & & \\ \operatorname{colog} \sqrt[3]{1{,}785} &= & & 8.91612 - 10 \\ \log (3.246)^3 &= 1.53405 & & \\ \operatorname{colog} (3.246)^3 &= & & 8.46595 - 10 \\ \log 8.963 &= 0.95245 & & \\ \operatorname{colog} 8.963 &= & & \underline{9.04755 - 10} \\ \log N &= & & 38.54814 - 40 \\ & & = & 8.54814 - 10 \\ & & N = & 0.03533. \end{aligned}$$

EXERCISE 6–7

Evaluate the following problems:

1. $\dfrac{(2{,}184)(0.165)(94.2)(0.51)}{(\sqrt{257})(\sqrt[3]{231})(0.62)(5{,}857)}$.
2. $\dfrac{(\sqrt{3{,}685})(\sqrt[3]{72{,}348})(0.121)(62.8)}{(\sqrt[3]{543})(\sqrt[3]{512})(0.00004)}$.
3. $\dfrac{(5.8673)(\pi)(0.3536)(16.969)}{(4.9026)(56{,}773)(0.0086735)}$.
4. $\dfrac{(\sqrt[4]{56{,}745})(\sqrt{3.1416})(3.695)}{(0.000697)(1.2384)(17.92)}$.
5. $\dfrac{(1.548 \times 10^{-10})(19{,}432.1)}{(1.783 \times 10^{8})(5.964)}$.

16. Natural logarithms. Our discussions so far have dealt with logarithms to the base 10. However, we have noted that there are other bases in use, especially natural logs that use the number 2.71828 (denoted by e) as a base.

These logarithms to the base e are used considerably in higher mathematics and occur frequently in equations in the electrical field.

Fundamentally, there is no difference between logarithms to the base 10 and logarithms to any other base, such as base e. Since a logarithm, from our definition, is an exponent or power, then evidently any base might be used for a set of tables of logs.

For example, the logarithm of 400 to the base 10 is 2.60206, which means that the base 10 raised to the 2.60206 power will give a value of 400. In symbol form these statements are written

$$\log_{10} 400 = 2.60206$$

or

$$10^{2.60206} = 400.$$

Now these statements might just as easily have been written with the value e as a base. The power, however, would not be the same since e, which is equal to 2.71828+, cannot be raised to the same power as would 10 in order to give the same value of 400. So, from our tables of natural logarithms, we find that:

$$\log_e 400 = 5.99146 \text{ or } \log_{2.71828} 400 = 5.99146$$

which means

$$e^{5.99146} = 400 \text{ or } 2.71828^{5.99146} = 400.$$

Comparing this with the logs to the base 10, it is evident that fundamentally there is no difference and that either base might be used in the solution of a given problem.

EXAMPLE 6–20. Perform the indicated multiplication by use of logs to base 10 and also to base e.

$$N = 75.6 \times 4.382.$$

Solution:

To base 10:	To base e:
$\log_{10} 75.6 = 1.87852$	$^*\log_e 75.6 = 4.32543$
$\log_{10} 4.382 = \underline{0.64167}$	$\log_e 4.382 = \underline{1.47751}$
$\log_{10} N = 2.52019$	$\log_e N = 5.80294$
$N = 331.2$	$N = 331.26$

It should be noted that the characteristic is given with all natural logarithms and that certain special precautions must be observed in getting the natural log of a number that is not given in the table. For example, to find the natural log of a number which is $\frac{1}{10}$ or 10 times a number whose log is given, it is necessary only to subtract from or add to the given log the natural logarithm of 10.

EXAMPLE 6–21. Find $\log_e 45.0$.

Solution: Since $\log_e 4.50 = 1.50408$, to find $\log_e 45.0$, add $\log_e 10$ to $\log_e 4.50$.

$$\begin{aligned} \log_e 4.50 &= 1.50408 \\ \log_e 10 &= \underline{2.30259} \\ \log_e 45.0 &= 3.80667. \end{aligned}$$

EXAMPLE 6–22. Find $\log_e 0.450$.

Solution: Since $\log_e 4.50 = 1.50408$, to find $\log_e 0.450$, subtract $\log_e 10$ from $\log_e 4.5$.

$$\begin{aligned} \log_e 4.50 &= 11.50408 - 10 \\ \log_e 10 &= \underline{\ \ 2.30259\qquad\ } \\ \log_e 0.45 &= \ \ 9.20149 - 10. \end{aligned}$$

EXAMPLE 6–23. Let us suppose that we want to get the logarithm of 4.5 to the base 10 and we already have the log to the base e.

Solution: $\log_e 4.5 = 1.50408.$

Now $\log_{10} 4.5 = x.$

So, rewriting these in exponent form,

$$e^{1.50408} = 4.5$$

$$10^x = 4.5.$$

Therefore, $e^{1.50408} = 10^x$

or $1.50408 \log_{10} e = x \log_{10} 10$

and $$x = \frac{1.50408 \log_{10} e}{\log_{10} 10}.$$

But $\log_{10} e = 0.4343$ and $\log_{10} 10 = 1$;

so $x = (1.50408)(0.4343) = 0.65322.$

---

* Some texts use the symbol, ln, to indicate $\log_e$.

Therefore, to change a log from base e to base 10, it is necessary only to multiply the given log by 0.4343, which is the log of e to the base 10.

Similarly, it may be shown that to change a log from base 10 to base e, it is necessary only to multiply the given value by 2.3026, which is the log of 10 to the base e, or the reciprocal of 0.4343.

As an example of the occurrence of natural logarithms in applied problems, consider the following equations:

The equation for the rise of current in a direct-current circuit containing inductance and resistance is

$$i = \frac{E}{R}\left[1 - e^{-\frac{Rt}{L}}\right] \text{ amperes,}$$

where i is the current, t seconds after the switch is closed.

$$\begin{aligned} E &= \text{impressed voltage.} \\ e &= \text{base of natural logs.} \\ R &= \text{resistance in ohms.} \\ L &= \text{inductance in henrys.} \end{aligned}$$

The equation for the fall of current in a direct-current circuit is

$$i = I_0 e^{-\frac{Rt}{L}},$$

where I_0 = initial current and other values the same as in the preceding equation.

The work W done by a gas expanding from a volume V_1 to a volume V_2 at a constant temperature is represented by the expression

$$W = p_1 V_1 \log_e \frac{V_2}{V_1},$$

where p_1 is pressure.

Example 6–24. In the equation

$$i = \frac{E}{R}\left[1 - e^{-\frac{Rt}{L}}\right]$$

solve for i when $E = 6\text{v}$; $R = 12$ ohms; $t = 0.03$ sec; and $L = 0.4$ h.

Solution: Substituting all values,

$$i = \frac{6}{12}\left[1 - e^{\frac{-(12)(0.03)}{(0.4)}}\right]$$

$$i = \frac{1}{2}\left[1 - \frac{1}{e^{\frac{0.36}{0.4}}}\right] \qquad \textit{Note: } e^{-\frac{Rt}{L}} = \frac{1}{e^{\frac{Rt}{L}}}.$$

$$i = \frac{1}{2}\left[1 - \frac{1}{e^{0.9}}\right].$$

Now $\qquad \log e$ to base e equals 1 or $\log_e e = 1$;

so $\qquad \log_e e^{0.9} = 0.9 \log_e e = (0.9)(1) = 0.9$

and $\qquad \log_{10} e^{0.9} = (0.9)(0.4343) = 0.39087.$

Then $\qquad e^{0.9} = 2.460$ (approx.).

Now
$$i = \frac{1}{2}\left[1 - \frac{1}{2.46}\right] = \frac{1}{2}(1 - 0.407),$$
$$i = \frac{1}{2}(0.593) = 0.2965 \text{ amp.}$$

Example 6–25. With the values of E, R, and L as given in Example 6–24, how long will it take the current to rise to 98% of its final value?

Solution: The final value of current is determined entirely by the value of R and can be computed by using $I = \frac{E}{R}$.

Thus,
$$I = \frac{6}{12} = 0.5 \text{ amp.}$$

Substituting in the equation for the increase of current we get:

$$(.98)(0.5) = \frac{6}{12}\left(1 - e^{-\frac{(12)t}{.4}}\right).$$

Then
$$.49 = \frac{1}{2}\left(1 - \frac{1}{e^{30t}}\right)$$

or
$$.98 = 1 - \frac{1}{e^{30t}}$$

and
$$\frac{1}{e^{30t}} = 1 - .98 = .02$$
$$e^{30t} = \frac{1}{.02} = 50$$
$$30t \log e = \log 50$$
$$t = \frac{\log 50}{30 \log e} = \frac{1.69897}{(30)(0.4343)} = 0.131 \text{ sec.}$$

17. Exponential equations. Very often we may encounter an equation in which the unknown appears as an exponent. Such an equation is the following:

$$5^x = 25.$$

It is very apparent in this equation that x must equal 2 because 5 must be squared to equal 25, but all such equations are not solved so easily.

In equations where the result cannot be written at sight, we must take logarithms of both sides of the equation and then use ordinary algebra methods for a final solution.

Example 6–26. Solve the following equation for x:

$$2^x = 40.$$

Solution: Taking the logarithm of each member, we have

$$\log 2^x = \log 40.$$

Then
$$x \log 2 = \log 40$$

and
$$x = \frac{\log 40}{\log 2} = \frac{1.60206}{0.30103} = 5.32.$$

Notice that the solution gives a value for x directly, not for $\log x$, which means that it is not necessary to find the antilog of 5.32 to get the final answer, because 5.32 is the final answer. The logarithms of 40 and 2 must be treated as ordinary numbers and not subtracted as in logarithmic division. Either logs to the base 10 or to the base e can be taken and the final answer will be the same.

To solve Example 6–26 entirely by logarithms we would proceed as follows:

$$\begin{array}{rl} \log\log 40 = \log 1.60206 = & 10.20468 - 10 \\ \log\log\ \ 2 = \log 0.30103 = & \ \ 9.47861 - 10 \\ & \underline{-\qquad\qquad +\ \ } \\ & \ \ 0.72607 \end{array}$$

$$\text{Then } x = \text{antilog } .72607 = 5.322.$$

(Note that a log log solution is required here.)

EXAMPLE 6–27. Solve the following equation for x and y:

$$2^{x+y} = 5.$$
$$3^y = 4^x.$$

Solution: Taking logs of both sides of each equation, we get

$$(x + y) \log 2 = \log 5$$

and
$$y \log 3 = x \log 4.$$

Then
$$x + y = \frac{\log 5}{\log 2} = \frac{0.69897}{0.30103} = 2.322$$

and
$$\frac{y}{x} = \frac{\log 4}{\log 3} = \frac{0.60206}{0.47712} = 1.2619.$$

Solving these two equations simultaneously by substituting $y = 1.2619x$ from the second equation into the first equation, we get

$$x + 1.2619x = 2.322$$
$$x = \frac{2.322}{2.2619} = 1.027$$

and
$$y = 2.322 - 1.027 = 1.295.$$

18. The slide rule. Logarithmic scales can be constructed by choosing a convenient unit and marking off lengths that will be proportional to the mantissas of the logarithms of numbers. Such a procedure is followed in the making of a slide rule, but the division points on the scale

are labeled with the numbers themselves instead of with the logarithms. The C and D scales of the common 10-in. rule are constructed on a scale 10 in. in length with the mantissas of the logarithms of the numbers from 1 to 10 laid off proportionally. The A and B scales are constructed in two equal sections, each 5 in. long. On each section the mantissas of the logarithms of numbers from 1 to 10 are also laid off proportionally. Therefore, each section of the A and B scales is half the length of the C and D scales. The divisions on each of the four scales are not equally spaced, because they are proportional to the mantissas of logarithms and the difference between the logarithms of successive numbers becomes smaller and smaller as the numbers get larger.

The A and D scales are located on the stationary part of the rule and the B and C scales on the sliding part of the rule. By moving the slide, the logarithms can be mechanically added or subtracted and hence multiplications and divisions are performed. The C and D scales are used for these operations because they give a greater degree of accuracy than the A and B scales. Also from our knowledge of logarithms, we know that the square of a number is found by multiplying the logarithm by 2, and the square root is found by dividing the logarithm by 2. It is evident then that the A and B scales, whose sections are half the length of the C and D scales, can be used to find the square or square root of a number.

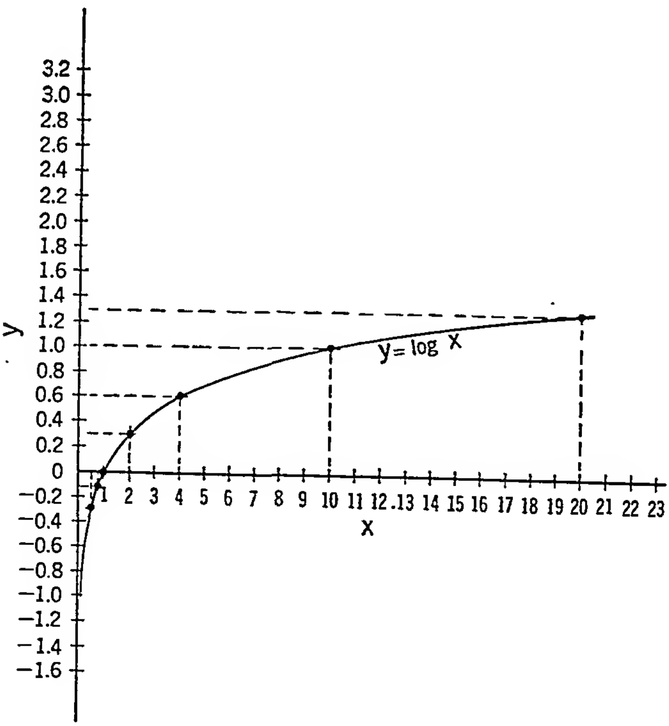


Fig. 6–1.

Finally, it is not necessary to look up the antilog in a table after using the slide rule, because the divisions are labeled with the antilogs and thus the answer to a problem is found directly on the rule.

19. Graph of $y = \log x$. In Fig. 6–1 is shown the graph of the equation $y = \log_{10} x$.

A study of this graph shows the following conditions:

1. There is no real logarithm for a negative number.

2. The logarithm of 1 is zero.

3. The logarithm of a positive number greater than 1 is positive and increases without limit as the number increases.

4. The logarithm of a positive number smaller than 1 is negative and decreases without limit as the number approaches zero.

The graph of $y = \log x$ could be drawn with any number as a base and it would take a path similar to that shown in Fig. 6–1. Such a graph is generally known as a semilogarithmic graph since only its vertical values are plotted to represent logarithms of numbers, whereas the horizontal values are plotted to a uniform scale.

A specially ruled paper has been devised to make it easy to plot these

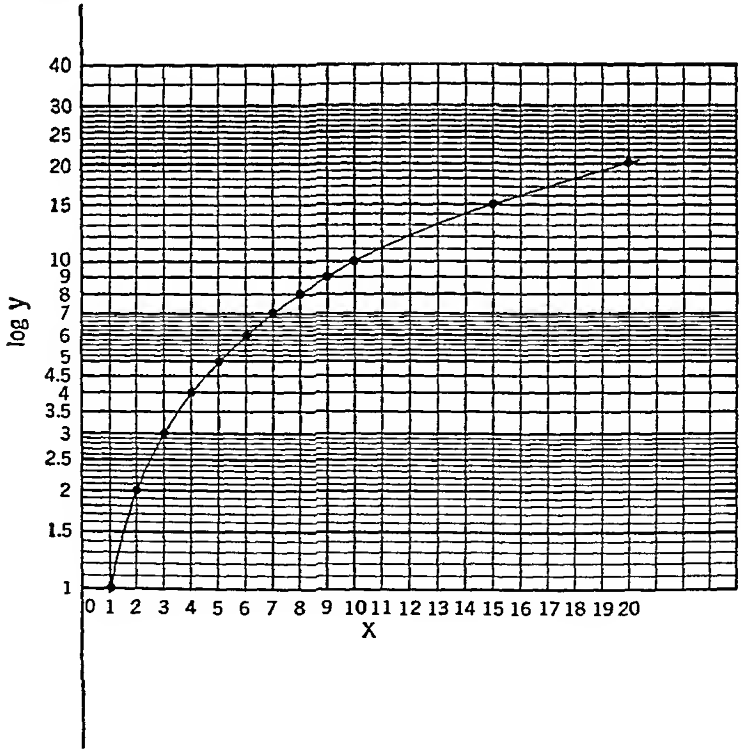


Fig. 6–2.

semilogarithmic graphs. Horizontally it has a uniform scale to represent values of x, whereas vertically it has a logarithmic scale to represent values of the logs of numbers. The number on the vertical scale at any point is a value for y but the height of this point above the horizontal axis is equal to $\log y$. Thus the distance up to the point marked 10 is actually equal to the log of 10; the distance up to the point marked 20 is equal to the log of 20; and so on. Therefore, by plotting a given table of values of x and y on this semilogarithmic paper, we automatically plot $\log y$ as a function of x, without looking up the logarithms. Fig. 6–2 shows a portion of a semilogarithmic graph sheet.

Semilogarithmic graphs are used in the plotting of equations where one of the variables is in the exponent. Such an equation is $2^y = x$, in which y is an exponent. This equation can be put in the form of a function of x by taking logarithms of both sides. Thus,

$$\log 2^y = \log x$$

or

$$y \log 2 = \log x$$

and

$$y = \frac{\log x}{\log 2}.$$

The graph of this equation will be similar to Fig. 6–1, since $\log 2$ is a constant value.

An interesting application of the semilogarithmic graph occurs in the *compound interest law.* If one quantity y varies with another quantity x so that its rate of increase or decrease is constantly proportional to its value, we have a condition exactly like an investment whose interest or depreciation is computed continuously at a fixed percentage rate. Such quantities vary according to the compound interest law and are represented by the equation $y = Pe^{rx}$, where P is the value of y when $x = 0$ and r is the fixed percentage rate. This equation when plotted on semilogarithmic paper produces a straight line and, for this reason, such graphs are much used in studying the growth of populations, bonded indebtedness, bank clearings, etc. Since *the graph becomes a straight line on semilogarithmic paper, only two points need to be determined in order to plot the graph.*

EXAMPLE 6–28. Plot the semilogarithmic graph of $y = 100e^{-0.4t}$.

Solution: When $t = 0$, $y = 100$; and when $t = 10$, $y = 100e^{-4} = 1.83$ (approx.).

The plotting of these two values gives the graph of Fig. 6–3. Further values of y that satisfy the given equation now can be read directly from the graph.

20. Logarithmic graphs. In certain kinds of statistical work, it is sometimes necessary to plot the logarithms of both variables, that is, to plot a graph to show how $\log y$ varies with $\log x$. This is done where very large and very small values of y and x are to be handled, because it cuts down the contrasts between the numbers. For example, it would be

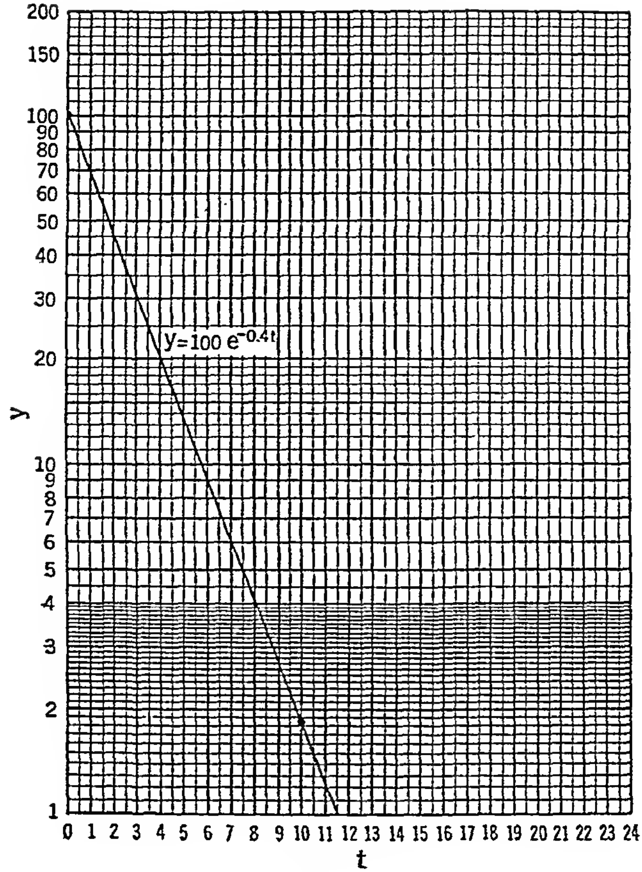


Fig. 6-3.

much easier to plot the log of 1,000,000 against the log of 0.0001 than to plot 1,000,000 against 0.0001, because log 1,000,000 is only 6, and log 0.0001 is -4, and the contrast between the logs is much less than between the numbers themselves.

Logarithmic graphs can be plotted on special logarithmic paper that is constructed so that it is not necessary to look up the logs of numbers. Both horizontal and vertical axes are laid off according to the mantissas of the logs of numbers. A point on the sheet for which $y = 30$ and $x = 12$ would have its actual distances from the reference axes equal to log 30 and log 12 respectively.

An application of the use of logarithmic graphs is presented in the equation $y = kx^n$, which is known as the *power law*. For this type equation the logarithmic graph is a straight line, because $\log y = \log K + n \log x$, which is an equation of the first degree. Therefore, only two points need to be found to determine the straight line.

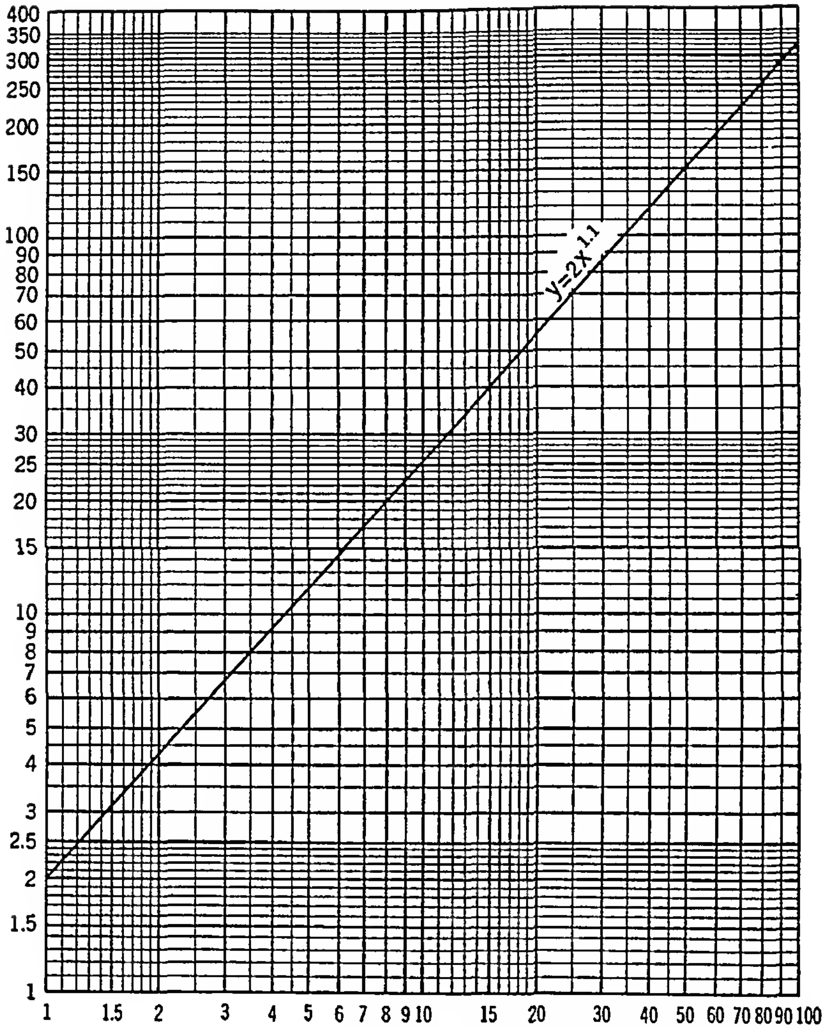


Fig. 6-4.

Example 6–29. Plot the logarithmic graph of the equation $y = 2x^{1.1}$.

Solution:

When $x = 1, \quad y = 2.$

When $x = 100, \quad y = 317$ (approx.).

The required graph is found by plotting these two points and drawing a straight line between them. Further values that satisfy now can be read directly off the graph which is shown in Fig. 6–4.

EXERCISE 6–8

1. If $\log_e 8.23 = 2.10779$, find $\log_e 8230$.
2. If $\log_e 645 = 6.46925$, find $\log_e 6.45$.

3. If $\log_e 3.62 = 1.28647$, find $\log_{10} 3.62$ and check with the tables to the base 10.

4. If $\log_{10} 292 = 2.46538$, find $\log_e 292$ and check with the tables to the base e.

Solve the following equations by the use of logarithms.

5. $3^x = 24$. 6. $9^x = 2$. 7. $2^{x^2} = 64$. 8. $5^{x+1} = 2^y$, $3^{x+y} = 4$.

9. Solve the following equation for t:

$$i = I_0 e^{-\frac{Rt}{L}}.$$

10. In Problem 9, find t when $I_0 = 0.5$ amp, $i = 0.25$ amp, $R = 12$ ohms, $L = 0.36$ h.

11. Solve the following equation for t:

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right).$$

12. In Problem 11, find t when $i = 0.12$ amp, $E = 6$ v, $R = 12$ ohms, and $L = 0.4$ h.

13. Solve the following equation for the ratio $\frac{x}{y}$:

$$\frac{\dfrac{x}{56.11} + \dfrac{y}{138.2}}{\dfrac{x}{56.11} + \dfrac{y}{69.10}} = \frac{30}{31}.$$

14. Solve the following for x and y:

$$5^{x-1} = 3^x,$$
$$4^x = 6^{x+y}.$$

15. Plot the semilogarithmic graph of the equation $y = 20e^{0.5x}$.

16. Plot the logarithmic graph of the equation $y = 8x^{2.4}$.

17. In photography a logarithmic curve called the $D \log E$ curve is used. This curve shows the variation of density D with the log of the exposure time E in seconds. The density is plotted along the vertical axis and $\log_{10} E$ is plotted along the horizontal axis. Plot on semilogarithmic paper a $D \log_{10} E$ curve from the following data:

E	D	E	D	E	D
0.00010	0.10	0.500	0.70	1,000	2.23
0.00020	0.10	1.000	0.85	2,000	2.30
0.00050	0.10	2.000	1.00	5,000	2.36
0.0010	0.11	5.000	1.20	10,000	2.39
0.0020	0.12	10.000	1.35	20,000	2.40
0.005	0.15	20.000	1.50	50,000	2.38
0.010	0.20	50.000	1.70	100,000	2.35
0.020	0.25	100	1.85	200,000	2.38
0.050	0.32	200	1.97	500,000	2.17
0.100	0.42	500	2.13	1,000,000	2.06
0.200	0.50				

EXERCISE 6-9

Solve each of the following problems by the use of logs:

1. Find the volume of a sphere with radius of 3.65 in.

$$V = \frac{4}{3}\pi R^3.$$

2. Find the radius of a sphere that contains 45.6 cu in.

3. What is the weight of one quart of water in grams? (Water weighs 62.5 lb per cubic foot.)

4. The following expression is used to find the area of a triangle when the three sides are known:

$$K = \sqrt{s(s-a)(s-b)(s-c)},$$

where K = area,
s = half the sum of the sides,
a, b, and c are the three sides.

Find the area of a triangle whose three sides are 20.72 in.; 15.33 in.; and 28.56 in., respectively.

5. The area of a circle is 1,973.6 sq in. Find the radius.

6. Using the formula for horsepower,

$$HP = \frac{PLAN}{33{,}000},$$

find HP when $P = 72$, $L = 2.5$, $A = 240$, $N = 120$.

7. Find the value of M from the formula

$$M = \frac{Wgl^3}{4bd^3B}$$

when $g = 980$, $W = 70$, $l = 40$, $b = 0.982$, $d = 0.586$, and $B = 0.0109$.

8. Find the value of n from the formula

$$n = \frac{360Lmgl}{\pi^2\theta r^4}$$

when $L = 59.4$, $m = 20$, $r = 0.322$, $g = 980$, $l = 24$, $\theta = 1.216$.

9. The expression for finding the total amount of money accumulated at the end of any period of time at compound interest is

$$A = P\left(1 + \frac{r}{K}\right)^{Kn},$$

where A = amount,
P = principal,
r = interest rate,
K = times per year compounded,
n = number of years.

Find the amount when \$150 is compounded quarterly at 3% interest for 10 yr.

10. Find the volume of a right circular cylinder, the radius of the base being 6.24 in. and height 2 ft 3.5 in. $V = \pi R^2 h$.

11. In tests on lubrication we have the formula

$$f = 20c\frac{\sqrt{v}}{p}.$$

Find f when $c = 0.0015$, $v = 200$, $p = 300$.

12. The diameter of a shaft for a certain engine is given by the equation

$$d = \sqrt[3]{\frac{D^2P^2R}{C}}.$$

Find d when $D = 13$ in., $P = 110$ lb, $R = 11$ in., and $C = 2{,}434$.

13. The expression for adiabatic compression is

$$V_2 = V_1\left(\frac{P_1}{P_2}\right)^{0.71}.$$

Find V_2 when $V_1 = 160$, $P_1 = 14$, and $P_2 = 80$.

14. The expression for flow of compressed air in pipes is

$$V = 58\sqrt{\frac{pd^5}{WL}}.$$

Find V when $p = 7$, $d = 11$ in., $W = 0.0672$, and $L = 45$.

15. The horsepower necessary to compress a gas in a single-stage air compressor is given by the formula

$$HP = \frac{144n}{(33{,}000)(n-1)}P_1V_1\left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right].$$

Find the horsepower required to compress 260 cu ft per minute from a pressure of 16 lb per square inch to 270 lb per square inch. Use $n = 1.25$.

16. In a circuit containing a capacitor C and a resistance R the instantaneous value of the charging current i at any time t is given by the equation

$$i = \frac{E}{R}\left(e^{-\frac{t}{Rc}}\right).$$

Find i when $E = 12$ v, $R = 9.5$ ohms, $c = 0.000029$ farad, and $t = 0.00002$ sec.

17. The equation for the quantity of water flowing over a contracted rectangular weir is shown by the following equation:

$$Q = \frac{10}{3}\left(b - \frac{2H}{10}\right)[(H+h)^{3/2} - h^{3/2}],$$

where Q is in cubic feet per second. Find Q when $b = 0.30$ ft, $H = 0.24$ ft, and $h = 0.0025$ ft.

18. The skin frictional resistance of the surface of a plane is given by the equation

$$R = 0.00000778L^{0.93}V^{1.86}b.$$

Find R when $L = 13$ ft, $V = 205$ ft per second, and $b = 3.25$ ft.

19. The maximum deflection of a beam of uniform cross section, loaded transversely and supported freely at the ends, is given by the equation

$$d = \frac{Wa}{3EIc}\left[\frac{b(a + c)}{3}\right]^{3/2}.$$

Find d when $W = 8{,}000$ lb; $E = 28.8 \times 10^6$; $I = 20.8$ in.^{4}; $a = 42$ in.; $b = 108$ in.; $c = 144$ in.

(*Note:* The dimension I is in in.^{4}, which means inches to the fourth power. Therefore, the 4 is a part of the unit and is not to be used in the equation.)

20. The following is a triode formula:

$$I_p + I_g = K\left[E_g + \frac{E_p}{\mu}\right]^{3/2},$$

where I_p is the plate current, I_g is the grid current, E_g is the grid voltage, E_p is the plate voltage, μ is the amplification factor, and K is a constant. Calculate $I_p + I_g$ if $K = 0.0005$, $E_g = 5$ v, $E_p = 240$ v, and $\mu = 8$.

21. The emission current for a heated filament is given by the equation

$$I = AT^2e^{-\frac{B}{T}}.$$

If $A = 60$ and $B = 52.4 \times 10^3$ for a tungsten filament, find I at a temperature of 2,500° K.

(*Note:* A Kelvin temperature is equal to temperature centigrade + 273.18.)

22. In a simple steam engine, the equation for the mean effective pressure is

$$P_m = P\left(\frac{1 + \log_e R}{R}\right) - p.$$

Determine P_m if $P = 105$ lb per square inch, $p = 14.9$ lb per square inch, and $R = 3.54$.

23. The flow of saturated steam through a nozzle is given by the empirical equation

$$W = 60\,\frac{Ap^{0.97}}{\sqrt{x}}.$$

Find W when $A = 0.17$, $p = 87.5$, and $x = 0.97$.

24. The temperature of a body cooled by moving air is given by the equation $T = T_0e^{-Kt}$, where T is the difference in temperature between the body and the air, T_0 is the initial temperature difference, t is the time in seconds for cooling, K is the radiation constant, and e is the base of natural logs. Find T when $T_0 = 35°$ C, $t = 15$ sec, and $K = 0.00145$.

25. The hysteresis loss in a transformer is computed from Steinmetz's empirical equation $W_h = nB^{1.6}$, where W_h is the loss in ergs per

cycle, n is a constant depending upon the material, and B is the maximum flux density in gausses. Determine W_h when $n = 0.013$ and $B = 12{,}500$ gausses.

26. The number of 30-w germicidal lamps required for a 30-ft-long room with a given number of occupants is found from the equation

$$N = \frac{M}{1530H_1C\frac{P_0}{B}}\left(\frac{H_1}{W}\right)^{0.4},$$

where N is the number of lamps, M is the number of occupants, H_1 is the depth of irradiated stratum, C is the circulation factor, $\frac{P_0}{B}$ is the relative number of bacteria per cubic foot, and W is the width of the room. Find N when $M = 40$; $H_1 = 5$; $C = 14.5$; $\frac{P_0}{B} = 0.44$; and $W = 30$ ft.

27. The ratio of the tension T_1 on the tight side of a belt to the tension T_2 on the loose side of a belt is expressed by the equation

$$\frac{T_1}{T_2} = e^{f\theta},$$

where e is the base of natural logs, f is the coefficient of friction, and θ is the angle of contact in radians. Find T_2 when $T_1 = 70$ lb, $f = 0.2$, and $\theta = \pi$ radians.

REVIEW EXERCISE 6–10

1. Heat loss in BTU per hour through a pipe is given by the equation,

$$Q = \frac{2\pi KL(T_1 - T_2)}{\log_e \frac{r_2}{r_1}}$$

where K is a constant determined by the material, L is the length of the pipe in feet, T_1 is the inside temperature, T_2 is the outside temperature, r_2 is the outside radius, and r_1 is the inside radius. Find Q when $K = 245$, $L = 60$ ft, $T_1 = 282°$ F, $T_2 = 267°$ F, $r_2 = 10$ in., and $r_1 = 9\frac{1}{4}$ in.

2. The length in feet of a freely hanging wire is found from the equation,

$$S = a(e^{\frac{x}{a}} - e^{-\frac{x}{a}}),$$

where a is a constant and x is one-half the span. Find the length of a wire hanging between two poles when $a = 2120$ and the poles are 360 ft apart. ($e = 2.71828$.)

3. A 180-lb body dropped out of an airplane will fall 1200 ft in 11 sec and will then have a constant velocity of 173 ft per sec. The equation for its velocity is

$$V = 173(1 - e^{-\frac{32.1}{173}t}),$$

where t is the time of falling in seconds and $e = 2.71828$. Find the velocity at the end of 5 seconds.

4. In Problem 3, find the time required to reach a velocity of 150 ft per sec.

5. When gas expands at a constant entropy, the pressure-volume equation is

$$P_1V_1^{1.406} = P_2V_2^{1.406}.$$

Find V_2 when $P_1 = 18.5$, $V_1 = 19.4$, and $P_2 = 35.8$.

6. The diameter of a connecting rod is determined by the equation,

$$d = 0.02758\sqrt{DL\sqrt{P}},$$

where d is the diameter of the rod in inches, D is the diameter of the engine cylinder in inches, L is the length of the rod, and P is the maximum steam pressure in pounds per square inch. Find d when $D = 30$ in., $L = 72$, and $P = 160$.

7. A solid cylindrical cast-iron column will be crushed by a weight (P) in pounds in accordance with the equation,

$$P = 9.89 \times 10^4 \frac{d^{3.55}}{L^{1.70}},$$

where d is the column diameter in inches and L is the column length in feet. Find P when $d = 5\frac{3}{8}$ in. and $L = 8.5$ ft.

8. The equation for the weight that will crush a wrought-iron column is

$$P = 2.996 \times 10^5 \frac{d^{3.55}}{l^2},$$

where d is the diameter in inches and l is the length in feet. Find P for a wrought-iron column with the same dimensions as in Problem 7.

9. A brass wire with a weight hung from its free end is stretched according to the equation,

$$S = \frac{mgL}{\pi r^2 K},$$

where m is the weight, $g = 980$, L is the length of the wire, r is the radius of the wire, and K is a constant. Find S when $m = 930.6$ gm, $L = 306$ cm, $r = 0.3$ cm, and $K = 1.285 \times 10^{10}$.

10. The approximate capacitance between two equal size wires of a transmission line is given by the equation,

$$C = \frac{1.94 \times 10^{-2}}{\log\left(\frac{S - r}{r}\right)},$$

where C is the capacitance in microfarads per mile, S is the distance

between the conductors, and r is the radius of the conductor. *S and r must be in the same units.* Find C when $S = 4.5$ ft and $r = 0.129$ in.

11. The charge on a capacitor at any instant (t) after voltage has been applied is given by the equation,

$$q = Q(1 - e^{-\frac{t}{CR}}),$$

where q is the instantaneous charge in coulombs, C is the capacitance in farads, R is the resistance of the circuit, and e is equal to 2.71828. Find the time required to produce a charge of 1.27×10^{-2} when $Q = 1.6 \times 10^{-2}$, and $R = 3000$.

12. The cubic feet of water discharged from a triangular weir is given by the equation,

$$q = \frac{8C}{15}\sqrt{2gH^5},$$

where q is the quantity of water in cubic feet, C is the constant 0.592, g is the gravitational acceleration 32.2 ft/sec/sec, and H is the water head in feet. Find q when $H = 0.5$ ft.

13. The equation for the weight (W) of a cubic foot of saturated steam in a boiler is

$$W = \frac{P^{0.941}}{3.3036 \times 10^2},$$

where P is the pressure in pounds per sq in. Find W when $P = 275$ lb per sq in.

14. The equation for money at compound interest is

$$A = P\left(1 + \frac{r}{K}\right)^{Kn}.$$

Solve for n and determine its value for $A = \$1749.83$, $P = \$1600.00$, $r = 2\%$, and $K = 2$.

15. Solve for x in the equation,

$$(2^{3x-1})(3^{x+1}) = (4^{2x-3})(6^{x+1}).$$

16. Solve for x:

$$e^x + e^{-x} = 2.$$

17. Solve for x:

$$\log (2 + x) + \log (x + 1) = 1.$$

18. In the geometrical progression,

$$S = \frac{ar^n - a}{r - 1},$$

solve for n in terms of a, r, and S.

THE GREEK ALPHABET

LETTER	SIGN Capital	SIGN Lower Case
Alpha	Α	α
Beta	Β	β
Gamma	Γ	γ
Delta	Δ	δ
Epsilon	Ε	ε
Zeta	Ζ	ζ
Eta	Η	η
Theta	Θ	θ
Iota	Ι	ι
Kappa	Κ	κ
Lambda	Λ	λ
Mu	Μ	μ
Nu	Ν	ν
Xi	Ξ	ξ
Omicron	Ο	ο
Pi	Π	π
Rho	Ρ	ρ
Sigma	Σ	σ
Tau	Τ	τ
Upsilon	Υ	υ
Phi	Φ	φ
Chi	Χ	χ
Psi	Ψ	ψ
Omega	Ω	ω

Chapter 1

ANGLES AND FUNCTIONS OF ANGLES

THE STUDY of angles and their functions is taken up in the science of trigonometry. This word is derived from two Greek words, *trigonon*, meaning "triangle," and *metria*, meaning "measurement." Therefore, trigonometry treats of the measurement of triangles.

A study of trigonometry and its methods is important to students of technical subjects because they will find its use necessary in the solution of many problems in alternating-current circuits, mechanics, surveying, and other technical courses.

1. Angles. An angle is defined in geometry as the opening formed between two straight lines that are drawn from the same starting point. This definition, however, is not general enough for trigonometry, where angles of any magnitude, either positive or negative, must be considered. A definition to fit the requirements of trigonometry can be stated as follows:

An angle may be considered as being generated by the rotation of a line segment about one of its ends. The original position of the segment is called the initial side and the final position is called the terminal side.

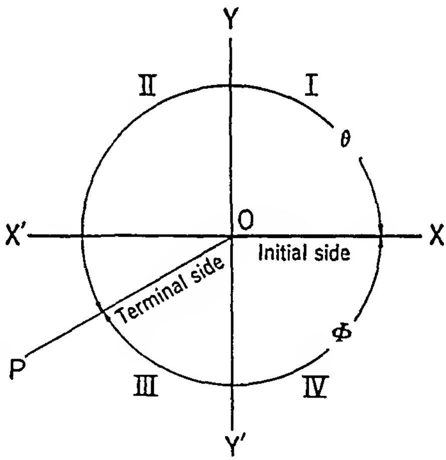


Fig. 1–1.

In Fig. 1–1, suppose we let a line OP start from the position OX and rotate in a counterclockwise direction about the origin, O, to a terminal position OP. Then the angle θ, indicated in the figure by the arrow, has been generated.

Now *if we define the counterclockwise rotation of the initial side as positive, then the angle θ in Fig. 1–1 is also positive.* But *if the rotation of the initial side is in a clockwise direction, the angle generated will be negative.* Thus, in Fig. 1–1 the angle Φ is considered as a negative angle.

In using a set of coördinate axes to generate angles as in Fig. 1–1, all angles begin with the line OX, the initial side. The angle generated is said to lie in the quadrant in which the terminal side lies. Thus, in Fig 1–1 the positive angle θ and the negative angle Φ each lie in quadrant III.

In Fig. 1–2a, a positive angle of $\theta°$ is generated by rotating the line OX in a counterclockwise direction to the position OP.

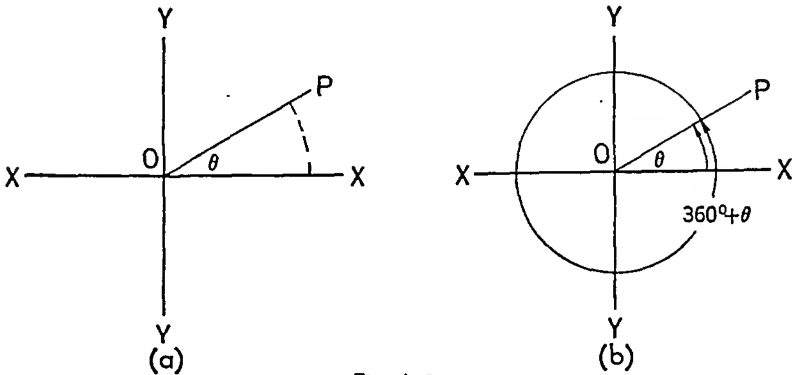


Fig. 1–2.

In Fig. 1–2b the generating line is rotated once completely about the origin and brought to rest in the same relative position as in Fig. 1–2a. However, since we have made one complete revolution plus the angle of Fig. 1–2a, we have generated 4 right angles plus $\theta°$, or $360° + \theta°$. Yet in each case we have the same terminal side. By a second full revolution of the generating line, we would get 8 right angles plus $\theta°$, or $720° + \theta°$. And this rotation might be continued indefinitely to give any number of angles having the same terminal side. So for n revolutions, the angle generated would be $n(360°) + \theta°$.

Likewise the generating line might be rotated in a clockwise direction, just as it was in a counterclockwise direction, and an indefinite number of negative angles obtained.

Evidently any given terminal position of the generating line will represent a series of positive angles and also a series of negative angles, each angle in the series differing from the next one to it by 1 revolution.

Example 1–1. With a protractor construct the following angles:

$+30°; +120°; +330°; -15°; -210°.$

Solution: See Fig. 1–3.

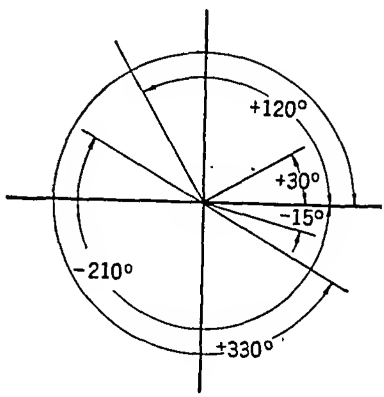


Fig. 1–3.

Now, it is possible to add or subtract two angles graphically. To add, the two angles should be placed in the same plane with a common vertex, with the initial side of the second angle placed on the terminal side of the first, and with each angle retaining its original direction. The sum of the two then will be found by measuring from the initial side of the first to the terminal side of the second. To subtract two angles, it is necessary only to add the second angle in the opposite direction to its original direction.

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Lambda	Λ	λ
Mu	Μ	μ
Nu	Ν	ν
Xi	Ξ	ξ
Omicron	Ο	ο
Pi	Π	π
Rho	Ρ	ρ
Sigma	Σ	σ
Tau	Τ	τ
Upsilon	Υ	υ
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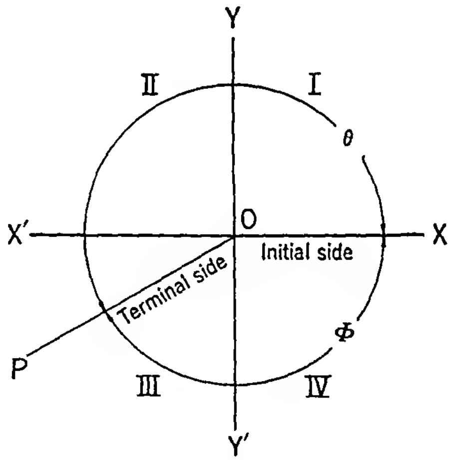

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If the angle terminates on one of the axes, it is called a *quadrantal angle* because it is not located in any particular quadrant.

The angle generated by rotating the initial side OX (Fig. 1–1), to the position OY is termed a right angle. By continuing the rotation of OX to the position OX' another right angle is generated, thus making two positive right angles from the original position OX. If OX is rotated to OY', a right angle is also generated but in the negative direction. Therefore, it can be seen that a right angle is formed by two lines perpendicular to each other. So, in rotating the line OX once completely around the origin and letting it come to rest directly on top of its starting position, there will be generated four right angles, either positive or negative depending upon the direction of rotation.

Since the rotation in either the positive or negative direction may be carried on indefinitely, it will be apparent that angles are not limited in extent. Thus the terminal side may be rotated any number of times about the origin and brought to a stop at the same point or some different point. The size of the angle is then determined by the number of complete revolutions about the origin plus the final distance from the origin.

2. Measurement of angles. There are two systems of measurement for angles in general use today: the *degree*, or *sexagesimal system*, and the *radian*, or *circular measure system*.

(a) *The degree system. A degree is defined as the ninetieth part of a right angle.* Thus, if a right angle is divided into 90 equal parts, each one of these equal parts is known as a degree. In turn each degree is divided into 60 equal parts and each of these divisions is known as a *minute*. A further division is made by dividing a minute into 60 equal parts, each of which is called a *second*. Thus we see that an angle is defined in terms of degrees, minutes, and seconds and that the following relationships are used:

60 seconds = 1 minute
60 minutes = 1 degree
90 degrees = 1 right angle

It is customary to use the following symbols to represent degrees, minutes, and seconds:

degrees = °
minutes = ′
seconds = ″

Therefore, 15 degrees, 25 minutes, and 40 seconds would be written 15° 25′ 40″. Notice that the symbols are all placed at the right and slightly above the figures.

Referring back to Fig. 1–1, it may be seen that each quadrant is a right angle, or 90 degrees; therefore, one complete revolution is equal to 4 right angles, or 360 degrees. *Any angle that is smaller than 90° is termed an acute angle. Any angle larger than 90° is termed an obtuse angle.*

In Fig. 1–2a, a positive angle of $\theta°$ is generated by rotating the line OX in a counterclockwise direction to the position OP.

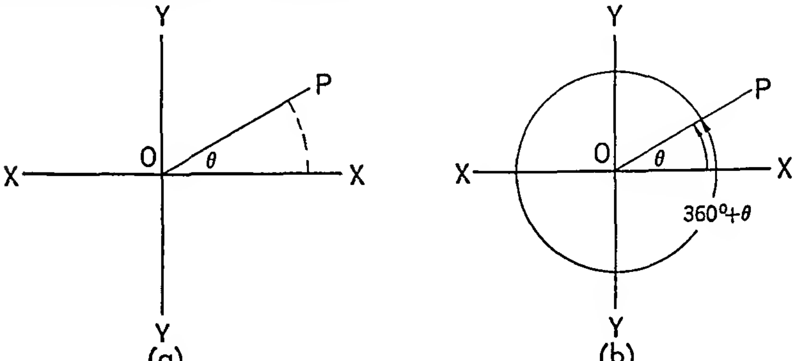

Fig. 1–2.

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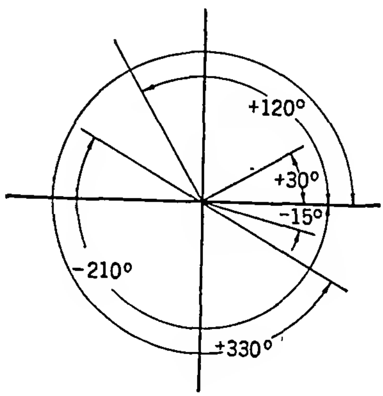

Fig. 1–3.

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Example 1–2. Add the following pairs of angles graphically:

(a) 30° and 120°;
(b) 45° and −30°.

Solution:

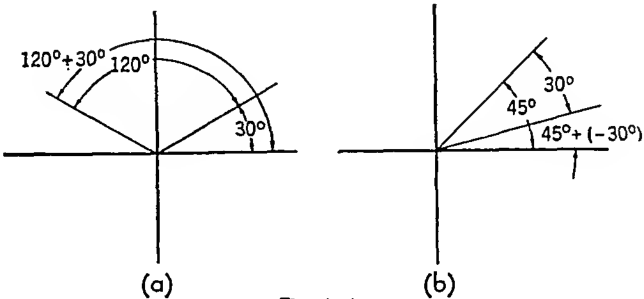


Fig. 1–4.

Example 1–3. Subtract graphically:

(a) 15° from 60°;
(b) −15° from +60°.

Solution:

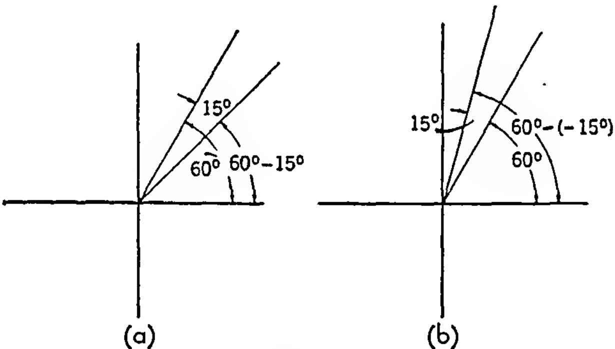


Fig. 1–5.

(b) *The radian system.* The other system of angular measurement is known as the radian or circular measure system.

A radian is defined as the angle at the center of a circle that intercepts an arc equal in length to the radius of the circle. Thus in Fig. 1–6 if the arc XM is equal in length to the radius OX, then, from the definition, the central angle XOM is equal to 1 radian.

If we wish to find the number of radians in some other angle such as XON, it is necessary only to find the number of times the radius is contained in the arc XN.

In considering the central angle and its arc, the angle XOM is said to subtend the arc XM. Thus, the following proportion results:

$$\frac{\text{angle } XON}{\text{angle } XOM} = \frac{\text{arc } XN}{\text{arc } XM}.$$

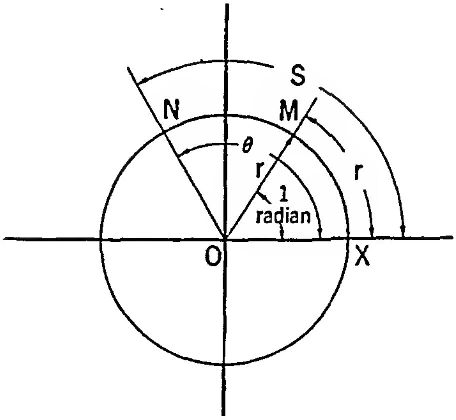


Fig. 1–6.

By giving the angle XON the symbol θ and its arc XN the symbol S, we may write the proportion like this:

$$\frac{\theta \text{ in radians}}{1 \text{ radian}} = \frac{S}{r}.$$

This is true because angle XOM equals 1 radian if arc XM equals r, the radius. Then

$$\theta = \frac{S}{r}, \quad \text{or} \quad \text{angle} = \frac{\text{arc}}{\text{radius}},$$

and

$$S = r\theta.$$

Therefore, the *length of an arc* is equal to the radius multiplied by the angle in radians.

Since the radius is contained in the circumference of the circle 2π times, it is evident that

$$2\pi \text{ radians} = 360°$$

and

$$\pi \text{ radians} = 180°.$$

From this we get

$$1 \text{ radian} = \frac{180°}{\pi} = 57.296° = 57°\,17'\,45''$$

and

$$1° = \frac{\pi}{180} \text{ radian} = 0.01745 \text{ radian}.$$

It is often convenient to convert from degrees to radians or from radians to degrees. When converting from degrees to radians, the angle at times may be expressed as a multiple of π radians, since π radians $= 180°$.

EXAMPLE 1–4. Express 120° in radians as a multiple of π.

Solution:

Since $$1° = \frac{\pi}{180} \text{ radian},$$

then $$120° = (120)\,\frac{\pi}{180} = \frac{120\pi}{180} = \frac{2}{3}\pi \text{ radians}.$$

In many trigonometry texts the word radians is omitted and 120° is said to be $\frac{2}{3}\pi$.

EXAMPLE 1–5. Convert 120° 20′ 30″ to radian measure.

Solution: First change the minutes and seconds to decimal parts of a degree. Thus,

since $$60' = 1° \text{ and } 3{,}600'' = 1°,$$

then $$20' = \frac{20}{60} = \frac{1}{3} = 0.33333°$$

and $$30'' = \frac{30}{3600} = \frac{1}{120} = 0.00833°.$$

To change 120° 20′ 30″ to radians, we know that 1° = 0.01745 radian. Then

$$\begin{aligned} 120° &= (120)(0.01745) = 2.09440 \text{ radians} \\ 20' &= (0.33333)(0.01745) = 0.00582 \text{ radian} \\ 30'' &= (0.00833)(0.01745) = \underline{0.00015} \text{ radian} \\ 120°\,20'\,30'' &= 2.10037 \text{ radians} \\ &= 2.1 \text{ radians (approx.)}. \end{aligned}$$

EXAMPLE 1–6. Convert 2.56 radians to degrees, minutes, and seconds.

Solution: Since 1 radian = 57.296°, 2.56 radians will equal in degrees (2.56)(57.296) = 146.67776°.

Now $0.67776° = (0.67776)(60) \text{ min} = 40'.6656$

and $0.6656' = (0.6656)(60) \text{ sec} = 39.936 \text{ or } 39''.9.$

So $2.56 \text{ radians} = 146°\,40'\,39''.9.$

Conversion tables have been formed for converting from degrees to radians or from radians to degrees and will be found in the appendix. These tables simplify and facilitate the work to a great extent.

EXAMPLE 1–7. Change 106° 15′ 16″ to radians by use of the tables.

Solution: From the tables

$$\begin{aligned} 106° &= 1.85005 \\ 15' &= 0.00436 \\ 16'' &= \underline{0.00008} \end{aligned}$$

Then $106°\,15'\,16'' = 1.85449.$

EXAMPLE 1–8. Change 2.432 radians to degrees, minutes, and seconds.

Solution:

$$\begin{aligned} 2.000 \text{ radians} &= (2)(57.2958) &&= 114.5916° \\ 0.400 \text{ radian} &= (0.4)(57.2958) &&= 22.9183° \\ 0.030 \text{ radian} &= (0.03)(57.2958) &&= 1.7189° \\ \underline{0.002 \text{ radian}} &= \underline{(0.002)(57.2958)} &&= \underline{0.1146°} \\ 2.432 \text{ radians} & &&= 139.3434° \end{aligned}$$

But $0.3434° = (0.3434)(60) = 20'.6040$

and $0.604' = (0.604)(60) = 36''.240.$

Then $2.432 \text{ radians} = 139°\,20'\,36''.24.$

We can now apply the formula, $s = r\theta$, which gives the length of an arc in terms of the radius of the circle and the angle subtended by the arc.

Example 1–9. Find the angle at the center of a circle subtending an arc of 12 in., the radius of the circle being 10 in.

Solution:
$$\theta = \frac{S}{r},$$
$$\theta = \frac{12}{10} = 1.2 \text{ radians;}$$
in degrees,
$$\theta = 68° \, 45' \, 18''.$$

EXERCISE 1–1

1. Construct the following angles with a protractor and state the quadrant in which each is located.

(a) 45°; −215°; 315°; −330°;
(b) −150°; 225°; −15°; 175°.

With a protractor add the following pairs of angles graphically and state the quadrant in which the total angle lies:

2. 45° and 135°.
3. 75° and 30°.
4. 90° and −30°.
5. −45° and 15°.

With a protractor subtract the following pairs of angles and state the quadrant in which the total angle lies:

6. 30° from 240°.
7. −120° from 330°.
8. 90° from −210°.
9. 90° from −90°.

Perform graphically the operations indicated in the following:

10. $90° + A_2$.
11. $270° - A_1$.
12. $360° + (-A_1)$.
13. $180° - (A_3)$.

(*Note:* A_1 represents a positive angle of first quadrant value, A_2 a positive angle of second quadrant value, A_3 a positive angle of third quadrant value, and A_4 a positive angle of fourth quadrant value.)

Convert each of the following angles to radian measure. Use multiples of π where convenient.

14. 30°.
15. 45°.
16. 210°.
17. 330°.
18. 15° 25′ 36″.
19. 110° 20′ 40″.
20. 300° 18′ 25″.
21. 221° 17′ 55″.

Convert each of the following angles to degree measure:

22. $\frac{3\pi}{8}$ radians.
23. 3.25 radians.
24. 1.58 radians.
25. $-\frac{5\pi}{6}$ radians.
26. $(2.178 - \pi)$ radians.
27. $\frac{3\pi}{5}$ radians.
28. 0.0675 radian.
29. $\frac{29}{64}$ radian.
30. 3.1416 radians.
31. $-\frac{2\pi}{3}$ radians.

32. An angle of 120° subtends an arc of 15 in. on the circumference of a circle. What is the radius of the circle?

33. The diameter of a graduated circle is 8 in. The graduations on the circumference are 20 min apart. What is the distance along the arc between any two successive graduations?

34. Find the length of an arc subtended by a central angle of 45° in a circle whose radius is 6 in.

35. The speed of a certain motor is given as 1,700 rpm. What will this be in radians per second?

36. What is the speed of a motor in revolutions per minute if its angular velocity is 58π radians per second?

37. A cast-iron flywheel has a maximum safe rim velocity of 100 ft per second. How many radians per second would this be for a flywheel 30 in. in diameter?

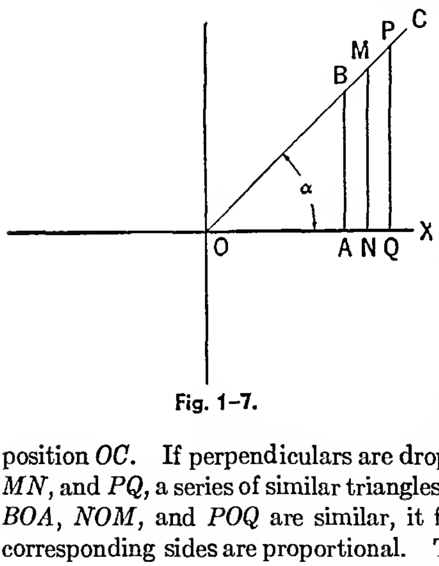


Fig. 1–7.

38. Through how many radians will the minute hand of a watch turn in 2 hr 45 min?

39. Through how many radians will the hour hand of a watch turn in 1 hr 12 min?

40. The speed of a generator is 3,600 rpm. What is this in radians per second?

3. The trigonometric ratios or functions. In Fig. 1–7, the angle α is constructed by rotating the initial side OX to the terminal position OC. If perpendiculars are dropped from OC upon OX, such as BA, MN, and PQ, a series of similar triangles will be formed. Since the triangles BOA, NOM, and POQ are similar, it follows from our geometry that the corresponding sides are proportional. Thus,

$$\frac{BA}{OB} = \frac{MN}{OM} = \frac{PQ}{OP};$$

also

$$\frac{OA}{OB} = \frac{ON}{OM} = \frac{OQ}{OP}$$

and

$$\frac{BA}{OA} = \frac{MN}{ON} = \frac{PQ}{OQ}.$$

That is to say, *these ratios remain unchanged as long as the angle does not change.* Since this is true, then each one of the above ratios is a function of the angle.

In addition to the three ratios shown, the reciprocal ratios also are true, thus making six ratios among the sides of a right triangle. The reciprocal ratios are as follows:

$$\frac{OB}{BA} = \frac{OM}{MN} = \frac{OP}{PQ},$$

$$\frac{OB}{OA} = \frac{OM}{ON} = \frac{OP}{OQ},$$

$$\frac{OA}{BA} = \frac{ON}{MN} = \frac{OQ}{PQ}.$$

Each one of these six ratios in a right triangle, since it is a constant quantity for any particular angle, is given a name as a function of the angle and these functions comprise the basis upon which the science of trigonometry is built.

It is customary in trigonometry to letter a triangle with the capital letters A, B, and C at the vertices denoting the respective angles, the letter C being at the right angle in the case of a right triangle, and the sides with the small letters, a, b, and c, as shown in Fig. 1–8. It should be apparent that a right triangle is one in which one of the angles is a 90° angle, or right angle. The side c opposite the right angle is called the *hypotenuse* and the sides a and b are called the *legs*. Consider the right triangle ABC of Fig. 1–8.

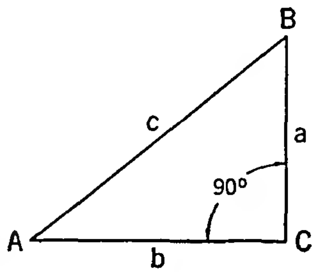


Fig. 1–8.

The names of the functions of angle A in the above triangle are as follows:

	Called	Written
$\frac{a}{c}$	sine of angle A	$\sin A$
$\frac{b}{c}$	cosine of angle A	$\cos A$
$\frac{a}{b}$	tangent of angle A	$\tan A$
$\frac{b}{a}$	cotangent of angle A	$\cot A$
$\frac{c}{b}$	secant of angle A	$\sec A$
$\frac{c}{a}$	cosecant of angle A	$\csc A$

A brief study of these six functions will show that they are only ratios of the sides of the triangle and that for any given value of angle A, there will be only one value for each of the functions.

There is one other relationship among the sides of the right triangle that is used considerably. This relationship is the one from geometry which says that in a right triangle, the square on the hypotenuse is equal to the sum of the squares on the two legs. In symbols this is written

$$c^2 = a^2 + b^2.$$

The following relationships will be apparent from our definitions and Fig. 1–8:

$$\sin A = \frac{a}{c} = \frac{\text{side opposite angle } A}{\text{hypotenuse}};$$

$$\cos A = \frac{b}{c} = \frac{\text{side adjacent to angle } A}{\text{hypotenuse}};$$

$$\tan A = \frac{a}{b} = \frac{\text{side opposite angle } A}{\text{side adjacent to angle } A};$$

$$\cot A = \frac{b}{a} = \frac{\text{side adjacent to angle } A}{\text{side opposite angle } A};$$

$$\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{side adjacent to angle } A};$$

$$\csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{side opposite angle } A}.$$

From these definitions it will be evident that approximate values of these functions may be determined by constructing a figure to a definite scale and using a protractor to measure the angle.

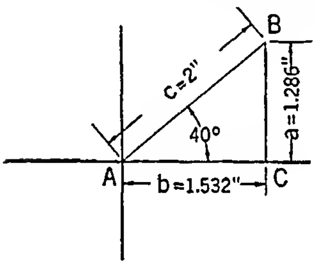


Fig. 1–9.

EXAMPLE 1–10. Determine the approximate values for the functions of 40°.

Solution:

With a protractor construct the angle CAB equal to 40°. Choose some suitable scale for the line AB, say 2 in. Then drop a perpendicular upon AC from the point B. Now measure the lengths of the lines BC and AC. Thus, BC = 1.286 in. and AC = 1.532 in.

From our definitions, we get

$$\sin A = \frac{a}{c} = \frac{1.286 \text{ in.}}{2 \text{ in.}} = 0.643;$$

$$\cos A = \frac{b}{c} = \frac{1.532 \text{ in.}}{2 \text{ in.}} = 0.766;$$

$$\tan A = \frac{a}{b} = \frac{1.286 \text{ in.}}{1.532 \text{ in.}} = 0.839;$$

$$\cot A = \frac{b}{a} = \frac{1.532 \text{ in.}}{1.286 \text{ in.}} = 1.192;$$

$$\sec A = \frac{c}{b} = \frac{2 \text{ in.}}{1.532 \text{ in.}} = 1.305;$$

$$\csc A = \frac{c}{a} = \frac{2 \text{ in.}}{1.286 \text{ in.}} = 1.556.$$

In a similar way the functions of any other angle could be found and a table giving the values for the functions of all angles could be made. But the values in such a table would be only approximate, at best, because of

the inherent inaccuracy in making the constructions and measurements. However, by advanced mathematical methods tables of varying degrees of accuracy have been prepared and will be discussed in a later article.

It should be noted that each function is an abstract value even though it is determined by dividing the length of one line by the length of another line. Thus,

$$\sin A = \frac{1.286 \text{ in.}}{2 \text{ in.}} = 0.643.$$

The value 0.643 is an abstract value and is not 0.643 in.

EXERCISE 1–2

1. Construct a right triangle with a 75° angle between the hypotenuse and the base and determine the six functions of 75° by measuring the lengths of the sides.

2. Construct a right triangle with a 60° angle between the hypotenuse and the altitude and determine the six functions of 60° by measuring the lengths of the sides.

3. From the values in Problems 1 and 2, determine which is larger:

(a) sin 60° or sin 75°.
(b) cos 60° or cos 75°.
(c) sin 60° or tan 60°.
(d) cos 75° or cot 75°.

4. Determine the six functions of angle A in the right triangle in which $a = b$, having given $c^2 = a^2 + b^2$.

5. Determine the six functions of angle A in the right triangle in which $b = 2a$, having given $c^2 = a^2 + b^2$.

6. Determine the six functions of angle A when $a = 3$, $b = 4$, and $c = 5$.

7. Determine the six functions of angle A when $b = 12$, $a = 5$, and $c = 13$.

8. Determine the length of side a in the right triangle in which $\sin A = \frac{3}{4}$ and $c = 8$.

9. Determine the length of side b in the triangle of Problem 8.

10. Determine the other functions of angle A in the triangle of Problem 8.

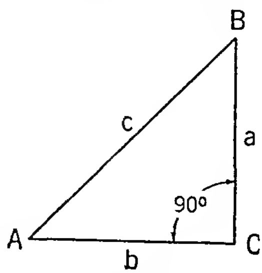


Fig. 1–10.

4. Functions of complementary angles. In geometry, we have learned that two angles are called complementary angles when their sum is 90°. Then in a right triangle, such as we have been studying, it is evident that the two acute angles must be complementary angles. Thus in Fig. 1–10 angle B is the complement of angle A, or $B = 90° - A$. (The sum of the

three angles of any triangle is 180° and the larger angle is always opposite the larger side.) In Fig. 1–10 the functions of angle A are

$$\sin A = \frac{a}{c}; \quad \tan A = \frac{a}{b}; \quad \sec A = \frac{c}{b};$$
$$\cos A = \frac{b}{c}; \quad \cot A = \frac{b}{a}; \quad \csc A = \frac{c}{a};$$

Also the functions of angle B may be seen to be

$$\sin B = \frac{b}{c}; \quad \tan B = \frac{b}{a}; \quad \sec B = \frac{c}{a};$$
$$\cos B = \frac{a}{c}; \quad \cot B = \frac{a}{b}; \quad \csc B = \frac{c}{b}.$$

An examination of these two sets of functions will show that

$$\sin A = \frac{a}{c} = \cos B; \quad \cot A = \frac{b}{a} = \tan B;$$
$$\cos A = \frac{b}{c} = \sin B; \quad \sec A = \frac{c}{b} = \csc B;$$
$$\tan A = \frac{a}{b} = \cot B; \quad \csc A = \frac{c}{a} = \sec B.$$

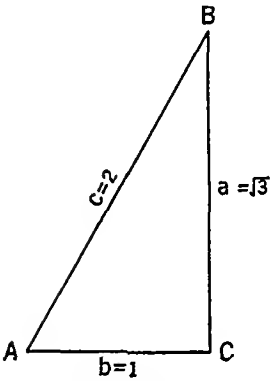


Fig. 1–11.

Therefore, it is clear that each function of an acute angle is equal to the cofunction of its complementary angle and that *cosine* means the complementary angle's sine and *cotangent* means the complementary angle's tangent.

EXAMPLE 1–11. Given that $c^2 = a^2 + b^2$ and $\sin A = \frac{\sqrt{3}}{2}$, find the other functions of A and also the functions of its complementary angle B.

Solution:

Since $$c^2 = a^2 + b^2$$

and $$\sin A = \frac{a}{c} = \frac{\sqrt{3}}{2},$$

then the other side, b, of the triangle may be found thus:

$$(2)^2 = (\sqrt{3})^2 + b^2, \quad 4 = 3 + b^2, \quad b^2 = 1, \quad b = 1.$$

Therefore,

$$\sin A = \frac{\sqrt{3}}{2} = \cos B, \quad \cot A = \frac{1}{\sqrt{3}} = \tan B,$$
$$\cos A = \frac{1}{2} = \sin B, \quad \sec A = 2 = \csc B,$$
$$\tan A = \sqrt{3} = \cot B, \quad \csc A = \frac{2}{\sqrt{3}} = \sec B.$$

5. Functions of 45°. The functions of certain special angles, among them 30°, 45°, and 60°, are easily found. For instance, in Fig. 1–12 let us take an isosceles right triangle. From our geometry we know that an isosceles triangle is one in which two of the sides are equal in length and therefore an isosceles right triangle is one in which the two legs are equal in length. In this figure angles A and B are equal to 45° and side a is equal to side b. Then since $c^2 = a^2 + b^2$ and $a = b$, we have

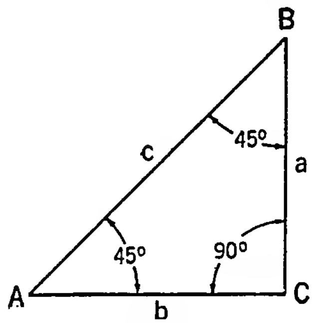


Fig. 1–12.

$$c^2 = a^2 + a^2 = 2a^2,$$

$$a^2 = \frac{c^2}{2},$$

$$a = \frac{c}{\sqrt{2}}.$$

Rationalizing,
$$a = \left(\frac{c}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{c\sqrt{2}}{2}.$$

Now

$$\sin 45° = \cos 45° = \frac{a}{c} = \frac{\frac{c\sqrt{2}}{2}}{c} = \frac{c\sqrt{2}}{2c} = \frac{\sqrt{2}}{2},$$

$$\tan 45° = \cot 45° = \frac{a}{b} = \frac{a}{a} = \frac{1}{1} = 1,$$

$$\sec 45° = \csc 45° = \frac{c}{a} = \left(\frac{2}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$= \frac{c}{\frac{c}{\sqrt{2}}} = (c)\left(\frac{\sqrt{2}}{c}\right) = \sqrt{2}.$$

Therefore, we have found the six functions of 45°. For convenience these six functions usually are changed into decimal form. Hence,

$$\sin 45° = \frac{\sqrt{2}}{2} = 0.7071;\ \tan 45° = 1;\ \sec 45° = \sqrt{2} = 1.414;$$

$$\cos 45° = \frac{\sqrt{2}}{2} = 0.7071;\ \cot 45° = 1;\ \csc 45° = \sqrt{2} = 1.414.$$

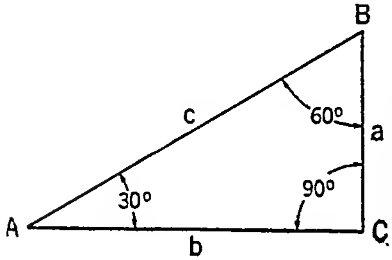


Fig. 1–13.

6. Functions of 30° and 60°. Let us now choose a triangle in which the two acute angles are 30° and 60°, respectively. In Fig. 1–13 a right triangle, with 30° and 60° angles, is drawn. In any right triangle in which the two acute angles are 30° and 60°, respectively, *the length of the leg opposite the*

30° angle is equal to half the length of the hypotenuse. Thus,

$$a = \frac{c}{2};$$

also, $$b = \sqrt{c^2 - a^2} = \sqrt{c^2 - \left(\frac{c}{2}\right)^2} = \sqrt{\frac{3c^2}{4}} = \frac{c}{2}\sqrt{3}.$$

Now the functions of 30° and 60° are

$$\sin 30° = \cos 60° = \frac{a}{c} = \frac{\frac{c}{2}}{c} = \left(\frac{c}{2}\right)\left(\frac{1}{c}\right) = \frac{1}{2};$$

$$\cos 30° = \sin 60° = \frac{b}{c} = \frac{\frac{c}{2}\sqrt{3}}{c} = \left(\frac{c\sqrt{3}}{2}\right)\left(\frac{1}{c}\right) = \frac{\sqrt{3}}{2};$$

$$\tan 30° = \cot 60° = \frac{a}{b} = \frac{\frac{c}{2}}{\frac{c}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3};$$

$$\cot 30° = \tan 60° = \frac{b}{a} = \frac{\frac{c}{2}\sqrt{3}}{\frac{c}{2}} = \sqrt{3}.$$

As in the case with a 45° angle the functions of the angles are usually put in decimal form for use. Thus,

$$\sin 30° = \cos 60° = 0.5;$$
$$\cos 30° = \sin 60° = 0.866;$$
$$\tan 30° = \cot 60° = 0.577;$$
$$\cot 30° = \tan 60° = 1.732.$$

These functions for 30°, 45°, and 60° can be utilized in finding the sides of triangles when one side is given and one of the acute angles is known to have one of these three values.

The following table gives a convenient way for remembering the sines and cosines of 0°, 30°, 45°, 60°, and 90°.

sin	0°	30°	45°	60°	90°
	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
cos	90°	60°	45°	30°	0°

EXAMPLE 1–12. In a right triangle in which one of the acute angles is 30°, find the hypotenuse and the other leg if the leg opposite the 30° angle is 6 in. long.

Solution: It is always advisable to construct a figure to approximate proportions for any problem. This will help to visualize the problem and make its solution simpler. Thus Fig. 1–14 is drawn with a $30°$ angle at A and the side BC drawn to represent 6 in.

Now $\quad \sin A = \dfrac{BC}{AB} = \dfrac{6}{AB}.$

But $\quad A = 30°.$

Therefore, $\quad \sin 30° = \dfrac{6}{AB}.$

and since $\quad \sin 30° = \dfrac{1}{2},$

$$\frac{1}{2} = \frac{6}{AB}.$$

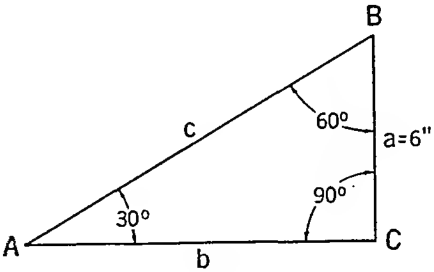


Fig. 1–14.

Then $\quad AB = (2)(6) = 12$ in.

Further $\quad \tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{6}{AC}.$

Then $\quad AC = 6\sqrt{3} = (6)(1.732) = 10.392.$

EXERCISE 1–3

1. Given that $c^2 = a^2 + b^2$ and that $\tan A = \dfrac{\sqrt{3}}{3}$, determine the other functions of angle A and also the functions of its complementary angle B.

2. Express each of the following functions as functions of an angle of less than $45°$:

$$\sin 75°;\ \tan 55°;\ \cos 80°;\ \cot 60°;\ \sin 90°.$$

3. Find the value of angle A in each of the following:

(a) $\sin A = \cos A.$
(b) $\sin A = \cos 4A.$
(c) $\tan (45° + A) = \cot A.$
(d) $\tan (90° - A) = \cot A.$

4. In a right triangle the acute angles are each $45°$. If the length of the hypotenuse is 10 in., what is the length of each of the legs?

5. The legs of a right triangle are equal to 8 in. and 13.856 in. respectively. Find the hypotenuse and the two acute angles.

6. In a right triangle in which the acute angles are respectively $30°$ and $60°$, the hypotenuse is found to be equal to 15 in. Find the length of each of the legs.

7. Functions as lines. Since the functions of an angle are ratios, they are also only numbers. Therefore, they may be represented by lines if we choose a unit of length such that the denominators of these ratios become equal to this chosen unit of length.

Thus, in Fig. 1–15 the radius of the circle is chosen as one unit and the angle θ constructed with perpendiculars drawn as shown. Then

$$OD = OB = OA = 1;$$

$$\sin\theta = \frac{ND}{OD} = \frac{ND}{1} = ND,$$

$$\cos\theta = \frac{ON}{OD} = \frac{ON}{1} = ON,$$

$$\tan\theta = \frac{ND}{ON} = \frac{BE}{OB} = \frac{BE}{1} = BE,$$

$$\cot\theta = \frac{ON}{ND} = \frac{OB}{BE} = \frac{AF}{OA} = \frac{AF}{1} = AF,$$

$$\sec\theta = \frac{OD}{ON} = \frac{OE}{OB} = \frac{OE}{1} = OE,$$

$$\csc\theta = \frac{OD}{ND} = \frac{OE}{BE} = \frac{OF}{OA} = \frac{OF}{1} = OF.$$

In each of the above cases, the fraction has finally been so arranged that the denominator is unity. In two of the functions, namely, the cotangent and the cosecant, it has been necessary to resort to a function of the complementary angle in order to make the denominator of the fraction unity.

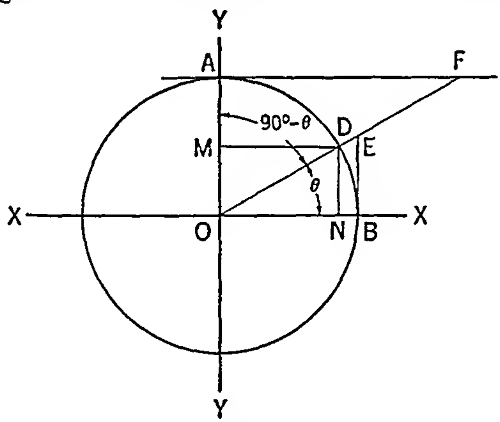


Fig. 1–15.

From Fig. 1–15 it is apparent that the sine of the complementary angle $(90° - \theta)$ is MD, which is equal to ON, which in turn is the cosine of angle θ. Also, the tangent of the complementary angle $(90° - \theta)$ is AF, which is also the cotangent of angle θ. Again, the secant of the complementary angle $(90° - \theta)$ is OF, which is also the cosecant of the angle θ. Therefore,

$$\tan(90° - \theta) = \cot\theta$$

and

$$\sec(90° - \theta) = \csc\theta.$$

But

$$\tan(90° - \theta) = \frac{AF}{OA}$$

and

$$\sec(90° - \theta) = \frac{OF}{OA}.$$

Therefore, $$\cot\theta = \frac{AF}{OA} = \frac{AF}{1} = AF$$

and $$\csc\theta = \frac{OF}{OA} = \frac{OF}{1} = OF.$$

8. Variation in the values of the trigonometric functions. In Fig. 1–16 a quarter circle with a unit radius is drawn. The angle θ is constructed with $\sin\theta$ equal to MP divided by OP, or equal to MP, since the radius of the circle has been chosen as unity. Now if we decrease the angle θ to

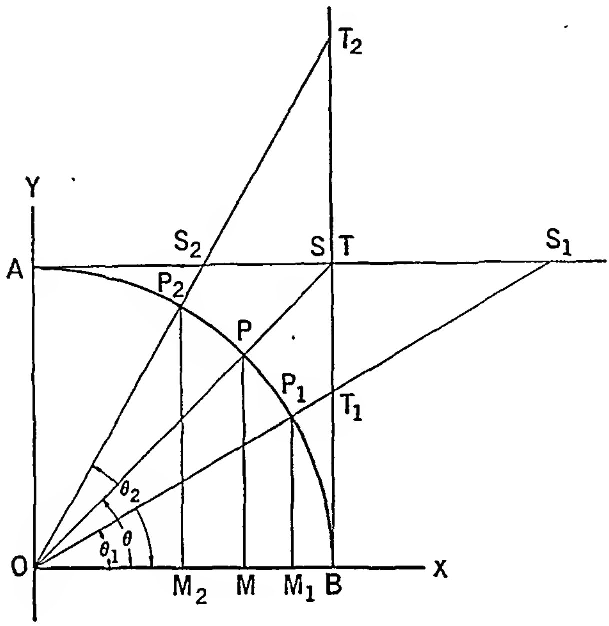


Fig. 1–16.

angle θ_1, then the sine will decrease to M_1P_1. Decreasing the angle further will also decrease the sine, so that when the angle is decreased to zero the sine is also zero. On the other hand, if the angle θ is increased to θ_2, then the sine will be increased to M_2P_2, and increasing the angle further toward 90° will increase the sine, until at 90° the sine becomes equal to OA, the radius of the circle, or unity. Hence, it is evident that, as the angle increases from 0 to 90°, the sine of the angle will increase from 0 to 1.

Again let us consider the tangent. For angle θ the tangent is BT. For the smaller angle θ_1, the tangent is BT_1, a smaller value, and as the angle decreases to 0, the tangent also decreases to 0. Now if the angle increases to θ_2, the tangent increases to BT_2. As the angle gets larger the tangent increases in size, until at 90° the terminal side of the angle coincides with OA and is parallel to BT_2. Since the terminal side is parallel to BT_2, it cannot cut BT_2 at any spot, and therefore the tangent for 90° must be infinitely large. This value is termed *infinity* and is written ∞, like the figure 8 turned on its side.

Similarly, the variations in the other functions may be determined. Hence, as the angle θ varies from 0 to 90°,

$$\begin{aligned} &\sin\theta \text{ varies from } 0 \text{ to } 1,\\ &\cos\theta \text{ varies from } 1 \text{ to } 0,\\ &\tan\theta \text{ varies from } 0 \text{ to } \infty,\\ &\cot\theta \text{ varies from } \infty \text{ to } 0,\\ &\sec\theta \text{ varies from } 1 \text{ to } \infty,\\ &\csc\theta \text{ varies from } \infty \text{ to } 1. \end{aligned}$$

From these it is evident that sines and cosines are never larger than 1. Tangents and cotangents may have any value from 0 to ∞. Secants and cosecants are never smaller than 1, but may be infinitely large.

For an angle of 0°, we have

$$\begin{aligned} \sin 0^\circ &= 0.\\ \cos 0^\circ &= 1.\\ \tan 0^\circ &= 0.\\ \cot 0^\circ &= \infty.\\ \sec 0^\circ &= 1.\\ \csc 0^\circ &= \infty. \end{aligned}$$

For an angle of 90°, we have

$$\begin{aligned} \sin 90^\circ &= 1.\\ \cos 90^\circ &= 0.\\ \tan 90^\circ &= \infty.\\ \cot 90^\circ &= 0.\\ \sec 90^\circ &= \infty.\\ \csc 90^\circ &= 1. \end{aligned}$$

9. Relations between the functions. When solving problems with trigonometric methods, it is often found convenient to change one function into another in order to arrive at an easier solution. In order to do this, it is advisable to become acquainted with the relationships among the various functions that give rise to the basic identities.

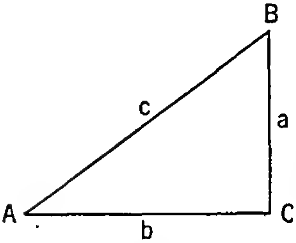


Fig. 1-17.

In Fig. 1-17 a conventional right triangle is drawn with the sides and angles lettered as usual.

$$\sin A = \frac{a}{c}, \qquad \tan A = \frac{a}{b}, \qquad \sec A = \frac{c}{b},$$

$$\cos A = \frac{b}{c}, \qquad \csc A = \frac{c}{a}, \qquad \cot A = \frac{b}{a}.$$

Since, from this triangle, we know that $c^2 = a^2 + b^2$, we may divide this equation through by c^2, thus getting

$$\frac{c^2}{c^2} = \frac{a^2}{c^2} + \frac{b^2}{c^2},$$

or

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Now $\frac{a}{c} = \sin A$ and $\frac{b}{c} = \cos A$.

Therefore,

$$\sin^2 A + \cos^2 A = 1. \tag{1}$$

It is customary to write $\sin^2 A$ for $(\sin A)^2$.

The student should note that $\sin^2 A$ or $(\sin A)^2$ does not mean the same thing as $\sin A^2$.

From this formula, which is one of the most important in trigonometry and should be studied carefully, we get the following:

$$\sin^2 A = 1 - \cos^2 A, \quad \text{or} \quad \sin A = \sqrt{1 - \cos^2 A};$$

and

$$\cos^2 A = 1 - \sin^2 A, \quad \text{or} \quad \cos A = \sqrt{1 - \sin^2 A}.$$

Again we may take the equation $c^2 = a^2 + b^2$ and divide it through by b^2 and get

$$\frac{c^2}{b^2} = \frac{a^2}{b^2} + \frac{b^2}{b^2},$$

or

$$\left(\frac{c}{b}\right)^2 = \left(\frac{a}{b}\right)^2 + 1.$$

Since

$$\frac{c}{b} = \sec A \text{ and } \frac{a}{b} = \tan A,$$

then

$$\sec^2 A = \tan^2 A + 1. \tag{2}$$

In like manner, dividing by a^2 will give

$$\frac{c^2}{a^2} = \frac{a^2}{a^2} + \frac{b^2}{a^2}.$$

Or

$$\csc^2 A = 1 + \cot^2 A. \tag{3}$$

From the original definitions of the functions, we know that

$$\sin A = \frac{a}{c} \text{ and } \cos A = \frac{b}{c}.$$

Then

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \left(\frac{a}{c}\right)\left(\frac{c}{b}\right) = \frac{a}{b} = \tan A.$$

Therefore,

$$\frac{\sin A}{\cos A} = \tan A. \tag{4}$$

Similarly,

$$\frac{\cos A}{\sin A} = \cot A. \tag{5}$$

Once more referring to the six functions of an angle it will be evident that

$$\sin A = \frac{1}{\csc A}, \qquad \cot A = \frac{1}{\tan A},$$

$$\cos A = \frac{1}{\sec A}, \qquad \sec A = \frac{1}{\cos A},$$

$$\tan A = \frac{1}{\cot A}, \qquad \csc A = \frac{1}{\sin A}.$$

In other words, the sine is the reciprocal of the cosecant, the cosine is the reciprocal of the secant, and the tangent is the reciprocal of the cotangent.

EXAMPLE 1–13. Show that the following identical equation is true:

$$\frac{1 + \tan^2 A}{1 + \cot^2 A} = \tan^2 A.$$

Solution: Changing $\tan^2 A$ to $\dfrac{\sin^2 A}{\cos^2 A}$ and $\cot^2 A$ to $\dfrac{\cos^2 A}{\sin^2 A}$ we get:

$$\frac{1 + \dfrac{\sin^2 A}{\cos^2 A}}{1 + \dfrac{\cos^2 A}{\sin^2 A}} = \frac{\sin^2 A}{\cos^2 A}.$$

Then, simplifying:

$$\frac{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}}{\dfrac{\sin^2 A + \cos^2 A}{\sin^2 A}} = \frac{\sin^2 A}{\cos^2 A},$$

$$\left(\frac{\cos^2 A + \sin^2 A}{\cos^2 A}\right)\left(\frac{\sin^2 A}{\sin^2 A + \cos^2 A}\right) = \frac{\sin^2 A}{\cos^2 A},$$

$$\frac{\sin^2 A}{\cos^2 A} = \frac{\sin^2 A}{\cos^2 A}.$$

10. Tables of functions. It has been shown in Article 3 that the functions of an angle are only ratios and that for any given angle the values of the functions will not change. Then it becomes evident that a complete table of values of the functions might be constructed. Such tables have been compiled showing the values of the four functions of sine, cosine, tangent, and cotangent, and are called *tables of natural functions.* Most tables do not give the values for the secant and cosecant, since their values can be determined from the cosine and sine respectively.

Let us turn to such a table giving the natural functions. The values for each of the four functions—sine, cosine, tangent, and cotangent—are given for every minute up to 90°.* Part of a sample page is shown in the accompanying table.

TABLE OF FUNCTIONS

24°

Min	Sin	Tan	Cot	Cos	Min
0	0.40674	0.44523	2.2460	0.91355	60
1	0.40700	0.44558	2.2443	0.91343	59
2	0.40727	0.44593	2.2425	0.91331	58
3	0.40753	0.44627	2.2408	0.91319	57
4	0.40780	0.44662	2.2390	0.91307	56
56	0.42156	0.46489	2.1510	0.90680	4
57	0.42183	0.46525	2.1494	0.90668	3
58	0.42209	0.46560	2.1478	0.90655	2
59	0.42235	0.46595	2.1461	0.90643	1
60	0.42262	0.46631	2.1445	0.90631	0
Min	Cos	Cot	Tan	Sin	Min

65°

* The trig tables in the back of the book are four-place for every 10′.

In order to read this table for angles at the top in degrees, the column headings at the top should be used. For angles at the bottom in degrees, the column headings at the bottom should be used. The minute column at the left is used with the degrees marked at the left of the table. The minute column at the right is used with the degrees marked at the right of the table.

EXAMPLE 1–14.

(a) Find the sine of 24° 3′; of 24° 58′.
(b) Find the sine of 65° 2′; of 65° 57′.

Solution:

(a) Since 24° is marked at the left and top, the column marked "Sin" at the top and the left-hand column of minutes are used. Thus,

$$\sin 24^\circ\ 3' = 0.40753$$

and

$$\sin 24^\circ\ 58' = 0.42209.$$

(b) Since 65° is marked at the right and bottom, then the column marked "Sin" at the bottom and the right-hand column of minutes are used. Thus,

$$\sin 65^\circ\ 2' = 0.90655$$

and

$$\sin 65^\circ\ 57' = 0.91319.$$

The student should notice that

the value 0.40753 is also cos 65° 57′ as well as sin 24° 3′;
the value 0.42209 is also cos 65° 2′ as well as sin 24° 58′;
the value 0.90655 is also cos 24° 58′ as well as sin 65° 2′
the value 0.91319 is also cos 24° 3′ as well as sin 65° 57′.

These must be true since we have shown that a function of an angle is equal to the cofunction of its complementary angle and an examination of the two angles in each of the above statements will show that they are complementary angles.

The same procedure as outlined in (a) and the relations of (b) of Example 1–13 for the sine and cosine is followed for finding the tangent and cotangent. Thus,

$$\tan 24^\circ\ 3' = 0.44627 = \cot 65^\circ\ 57'$$

and

$$\tan 65^\circ\ 2' = 2.1478 = \cot 24^\circ\ 58'.$$

Because of this relationship between the functions and cofunctions of complementary angles, the table reads only as far as 45° at the top. The values of functions for angles from 45° to 90° will be found by using the column headings at the bottom and the degrees at the bottom. It will be shown later that the functions of angles in the second, third, and fourth quadrants may be obtained also from trigonometric tables.

Often it is desired to find an angle when the function is given. This of course is the reverse of the process outlined under Example 1–14.

EXAMPLE 1–15.

(a) Find the angle whose tangent is 2.1478.
(b) Find the angle whose cotangent is 2.2408.

Solution: The value 2.1478 will be found in the column headed by tangent at the bottom and opposite the 2 in the minute column at the right. This, then, gives an angle of 65° 2′.

The value 2.2408 will be found in the column headed cot at the top and opposite the 3 in the minute column at the left. Therefore, the angle is 24° 3′.

It should be noted that 2.1478 is also the cotangent of 24° 58′, and 2.2408 is also the tangent of 65° 57′.

11. Interpolation. Often the angle for which a function is desired will have a decimal part of the minutes or seconds given, and in order to find the value of the desired function it becomes necessary to use the process of interpolation. This process is the same as that discussed in Chapter 6 of Part I under Logarithms.

EXAMPLE 1–16. Find sin 35° 15′.8.

Solution: From the four-place tables we get

$$\begin{aligned} \sin 35^\circ\ 20' &= 0.5783 \\ \sin 35^\circ\ 10' &= \underline{0.5760} \\ \text{difference} &= 0.0023. \end{aligned}$$

Since 15′.8 is 5.8/10 of the way between 10 min and 20 min and since the value of the sine increases as the angle increases, sin 15′.8 will be 5.8/10 of the way between 10′ and 20′.

Therefore, adding 5.8/10 of the difference to the value for 35° 10′ should give the value for 35° 15′.8. Thus,

$$\frac{5.8}{10} \times 0.0023 = 0.001334 \text{ or } 0.0013$$

Therefore, $\sin 35^\circ\ 15'.8 = 0.5760 + 0.0013 = 0.5773$.

EXAMPLE 1–17. Find the angle whose cosine = 0.6404.

Solution: From the tables,

$$\begin{aligned} \cos 50^\circ\ 10' &= 0.6406 \\ \cos 50^\circ\ 20' &= \underline{0.6383} \\ \text{difference} &= 0.0023. \end{aligned}$$

Since the cosine is decreasing as the angle is increasing, we must subtract our value from cos 50° 10′ and find what part this is of the total difference. Thus,

$$\begin{array}{r} 0.6406 \\ \underline{0.6404} \\ 0.0002 \end{array} \qquad \frac{0.0002}{0.0023} = 0.09.$$

This is 0.09 of 10′ or 0.9′. Therefore, 0.6404 = cos 50° 10.9′ or cos 50° 10′ 54″.

It should be noted that the sine and tangent increase, whereas the cosine and the cotangent decrease as the angle increases from 0° to 90°. Therefore, interpolation for the sine and tangent should be performed according to Example 1–16, and interpolation for the cosine and cotangent should follow the procedure in Example 1–17.

Strictly speaking, it is not correct to say that the change in a function of an angle is directly proportional to the change in the angle. The sine, for instance, changes from a value of zero to a value of unity as the angle changes from zero to 90°, or $\frac{\pi}{2}$ radians, which is equivalent to a change from zero to 1.5708, and the two ranges are not exactly the same. However, for small changes in the angle, such as tenths of a minute, the change in the function is nearly proportional to the angle and for all practical purposes is accurate. Therefore, the process of interpolation which assumes direct variation between the function and the angle is used and, practically speaking, gives as much accuracy as that of tables being used.

EXERCISE 1–4

1. Represent by lines in a circle with a unit radius the functions of 30°.
2. Represent by lines in a circle with a unit radius the functions of 45°.
3. Construct the angle θ where $\tan \theta = 2 + \sqrt{3}$.
4. Construct the angle θ where $\cos \theta = \frac{\sqrt{3.732}}{2}$.
5. Show proof that doubling any angle does not also double the sine.
6. Show proof that bisecting any angle does not also bisect the tangent.
7. Copy the table below and fill in the blank spaces from a table of natural functions.

θ	$\sin \theta$	$\cos \theta$	$\cot \theta$	$\tan \theta$
	0.9858			
12° 35′				
				11.430
			8.5989	
	0.2650			
		0.7071		
			1.1565	
		0.9997		
75° 15′				
				0.0096

Find the values of the functions for the following angles:

8. $25° 17'.8$.
9. $83° 5' 48''$.
10. $45° 45' 45''$.
11. $0° 17' 25''$.
12. $15° 15'.6$.

·Find the angles whose functions are given below:

13. $\sin \theta = 0.7854$.
14. $\cos \theta = 0.2187$.
15. $\tan \theta = 3.7283$.
16. $\sin \theta = 0.0148$.
17. $\cos \theta = 0.1796$.
18. $\tan \theta = 0.0487$.
19. $\cot \theta = 0.3457$.
20. $\cot \theta = 4.8257$.

Find the functions of the following angles:

21. $17° 18'.3$.
22. $79° 59'.9$.
23. $0° 4'.6$.
24. $50° 14'.7$.

Find the angles whose functions are given:

25. $\tan A = 3.4564$.
26. $\sin A = 0.9987$.
27. $\cos A = 0.4876$.
28. $\tan A = 0.1179$.

Prove that the following are true:

29. $\sin A = \sqrt{\dfrac{\tan^2 A}{\tan^2 A + 1}}$.

30. $\cos A = \sqrt{\dfrac{\cot^2 A}{1 + \cot^2 A}}$.

31. The diameter of a certain circle is 6 in. The circumference is graduated into parts $\frac{1}{4}$ in. long. What is the angle between any two successive points on the circumference? Express your answer in degrees, minutes, and seconds.

32. (a) In a right triangle, the sine of one angle is $\frac{2}{3}$. What are the values of the other functions of this angle in common fraction form? (b) If the hypotenuse of the above triangle is 10, what are the other sides?

REVIEW EXERCISE 1–5

Convert each of the following angles to radian measure.

1. $21° 20'.6$.
2. $72° 18'.4$.
3. $43° 45' 18''$.
4. $86° 37'.2$.
5. $65° 50' 30''$.
6. $114° 25'.3$.
7. $169° 47' 48''$.
8. $125° 15'.7$.
9. $138° 45' 23''$.
10. $147° 30'$.
11. $195° 27'.4$.
12. $265° 33' 18''$.
13. $210° 18'.2$.
14. $225° 45' 35''$.
15. $242° 57'.5$.
16. $303° 25'.4$.
17. $272° 41'.7$.
18. $350° 17' 42''$.
19. $289° 52'.4$.
20. $322° 41' 25''$.

Convert each of the following from radian to degree measure.

21. 0.15725.
22. 1.48970.
23. 0.85632.
24. 1.35207.
25. 0.96660.
26. 1.63275.
27. 1.93628.
28. 2.36418.
29. 2.85763.
30. 3.04326.
31. 3.17875.
32. 3.23654.
33. 3.97212.
34. 4.71238.
35. 4.38563.
36. 4.94731.
37. 5.00743.
38. 5.84115.
39. 6.28318.
40. 6.05716.

From the tables, find the four functions of each of the following:

41. 22° 14′.3.
42. 82° 25′.6.
43. 31° 41′.7.
44. 43° 23′.6.
45. 59° 59′.5.
46. 63° 52′.8.
47. 88° 31′.2.
48. 75° 5′.9.
49. 48° 29′.1.
50. 10° 10′.4.

From the tables, find the angles corresponding to the following functions:

51. $\sin \theta = 0.0375$.
52. $\sin \theta = 0.7856$.
53. $\sin \theta = 0.7071$.
54. $\cos \theta = 0.2588$.
55. $\cos \theta = 0.9988$.
56. $\cos \theta = 0.6372$.
57. $\tan \theta = 0.0582$.
58. $\tan \theta = 0.3758$.
59. $\tan \theta = 2.4503$.
60. $\cot \theta = 3.9307$.
61. $\cot \theta = 0.0411$.
62. $\cot \theta = 0.6319$.

63. Prove that $(\tan^2 x + 1) \sin^2 x = \tan^2 x$.

64. Prove that $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$.

65. Prove that $\dfrac{1 - \cos^2 x}{\cos^2 x} = \dfrac{1}{\cot^2 x}$.

66. Prove that $\sin x = \dfrac{1}{\sec x \cot x}$.

67. Prove that $\sec x \csc x = \cot x + \tan x$.

68. Determine the length of side b in the right triangle in which $\sin A = \dfrac{2}{3}$ and $c = 10.5$.

69. Determine the length of side a in Problem 68.

70. Determine the other functions of angle A and the functions of angle B in Problem 68.

71. The hypotenuse of a right triangle is equal to 2.5 in. One of the acute angles is 40° 15′.6. Find the two sides and the other acute angle.

72. The radius of the earth is approximately 3960 mi. Find the number of feet in an arc of 1° on any meridian.

73. A circle has a diameter of 8.5 in. The circumference is graduated in parts $\frac{3}{8}$ in. long. What is the angle between any two successive graduations?

74. What is the angular velocity of the second hand of a watch in radians per sec?

75. The moon has a diameter of 2,160 miles and its mean distance from the earth is 239,000 miles. What angle will the diameter subtend at the earth's surface?

76. A locomotive has a driving wheel 8 ft in diameter. If the wheel makes 200 rpm, what will be the speed of the train in miles per hour? What is the angular velocity of the driving wheel in radians per second?

77. A circle has a diameter of 20.92 inches. Find the length of an arc of the circumference cut off by a central angle of $168° 47'.6$.

78. What will a speed of 1,175 rpm represent in radians per second?

79. The two sides of a right triangle are 6.5 in. and 8.25 in. respectively. Determine the hypotenuse and the angles of the triangle.

80. Find the value of angle A in each of the following:

$$\begin{aligned} \sin 3A &= \cos A, \\ \cos 5A &= \sin A, \\ \tan A &= \cot 2A, \\ \cot A &= \tan 4A. \end{aligned}$$

Chapter 2

THE RIGHT TRIANGLE

1. The solution of the right triangle. In Chapter 1, the functions of angles were discussed, and it was shown how these functions are only ratios among the sides of a right triangle. Now it is possible, by means of these functions, to solve any right triangle completely.

Any right triangle has six parts — that is, three sides and three angles. Since it is a right triangle, one of the angles (the right angle) is always known to be 90°, and if one side is given also the other two sides and the third angle can be determined by use of the proper functions.

There are three conditions that exist whereby the right triangle may be solved. These conditions are as follows:

A. Given an acute angle and a side.
B. Given an acute angle and the hypotenuse.
C. Given two sides. (This condition may involve the two legs or one leg and the hypotenuse.)

A. *Given an acute angle and a side.*

EXAMPLE 2–1. In the right triangle of Fig. 2–1 solve for all the other parts, having given angle $A = 40°$ and side $b = 6.45$ in.

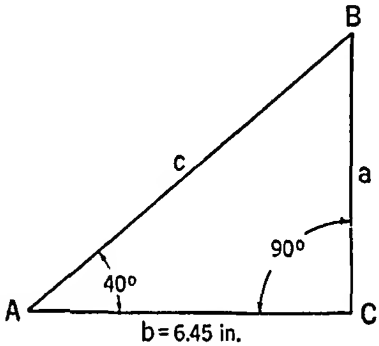


Fig. 2–1.

Solution:

Since $\angle A + \angle B + \angle C = 180°$ and $\begin{cases} \angle A = 40° \\ \angle C = 90°, \end{cases}$

then $\angle B = 180° - (90° + 40°) = 180° - 130° = 50°.$

The symbol $\angle$ is often used to denote an angle. Thus, $\angle A$ is read "angle A."

Now $\cos A = \dfrac{b}{c}.$

Therefore, $c = \dfrac{b}{\cos A} = \dfrac{6.45}{\cos 40°} = \dfrac{6.45}{0.7660}.$

This division performed on the slide rule gives

$$c = 8.42 \text{ in. (approx.)}.$$

Performed by logs,

$$
\begin{aligned}
\log 6.45 &= 10.8096 - 10 \\
\log 0.766 &= 9.8843 - 10 \\
&\qquad\qquad + \quad - \\
\log c &= 0.9253. \\
c &= 8.42 \text{ in.}
\end{aligned}
$$

To solve for side a we can use $\sin A$, since we have obtained side c, or we may use $\tan A$. This latter method is preferable since it uses only values given in the original problem and eliminates any error that may have resulted in solving for side c.

$$\tan A = \frac{a}{b}.$$

Therefore, $a = b \tan A = (6.45)(\tan 40°) = (6.45)(0.8391)$.

Performed on the slide rule, $a = 5.415$ (approx.).

Performed by logs,

$$
\begin{aligned}
\log 6.45 &= 0.8096 \\
\log 0.8391 &= 9.9238 - 10 \\
\log a &= 10.7334 - 10. \\
a &= 5.412.
\end{aligned}
$$

If the logarithmic solution is employed, it is not necessary to look up the values for $\cos 40°$ and $\tan 40°$ and then look up the logarithms of these values as we did in the case above, because there is a table giving the logarithms of the trigonometric functions directly. Thus, from these tables

$$\log \cos 40° = 9.8843 - 10$$

and

$$\log \tan 40° = 9.9238 - 10.$$

It must be remembered that the sine and cosine vary in value from 0 to 1, and consequently the characteristic of the logarithm of such functions will be negative. The same holds true for the tangent of an angle less than 45° and for the cotangent of an angle between 45° and 90°. Therefore, to all such logarithms in the tables the number -10 should be appended. Since the tangent of angles between 45° and 90° and the cotangent of angles less than 45° are greater than 1, the characteristic of the logarithm will be positive for these cases and -10 need not be appended to the logs in the tables unless a characteristic of $+10$ or larger is given.

Should there be need for interpolation in looking up the logs of trigonometric functions, it may be carried out in exactly the same way as was done in Chapter 1 for the values of the functions themselves.

The solution for sides c and a in Example 2–1 is as follows:

$$\cos A = \frac{b}{c},$$

or

$$c = \frac{b}{\cos A} = \frac{6.45}{\cos 40°}.$$

$$\begin{aligned}
\log 6.45 &= 10.8096 - 10 \\
\log \cos 40^\circ &= \underline{\ 9.8843 - 10} \\
\log c &= 0.9253. \\
c &= 8.42.
\end{aligned}$$

$$\tan A = \frac{a}{b},$$

or

$$a = b \tan A = (6.45)(\tan 40^\circ).$$

$$\begin{aligned}
\log 6.45 &= 0.8096 \\
\log \tan 40^\circ &= \underline{\ 9.9238 - 10} \\
\log a &= 10.7334 - 10 \\
\log a &= 0.7334. \\
a &= 5.412 \text{ in.}
\end{aligned}$$

EXERCISE 2-1

Solve the following right triangles for all unknown parts:

1. $B = 55^\circ$; $a = 58.3$.
2. $A = 43^\circ 28'$; $a = 2.38$.
3. $A = 76^\circ 16'$; $b = 32.86$.
4. $B = 18^\circ 15'$; $b = 16.83$.
5. $B = 68^\circ 33'.6$; $a = 2.4378$.
6. $A = 34^\circ 17'.8$; $a = 75.831$.
7. $B = 26^\circ 14'.4$; $b = 0.3826$.
8. $A = 62^\circ 15'.7$; $b = 0.1857$.
9. $B = 38^\circ 52' 30''$; $a = 0.0145$.
10. $A = 30^\circ$; $a = \sqrt{3}$.

B. *Given an acute angle and the hypotenuse.*

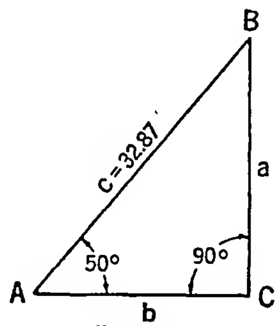


Fig. 2-2.

EXAMPLE 2-2. In the right triangle of Fig. 2-2 solve for all the other parts having given angle $A = 50^\circ$ and the hypotenuse $c = 32.87$.

Solution:

$$\begin{aligned}
\angle B &= 180^\circ - (A + C) \\
&= 180^\circ - (50^\circ + 90^\circ) = 180^\circ - 140^\circ = 40^\circ.
\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

Therefore, $a = c \sin A$.

By slide rule:

$$a = (32.87)(\sin 50^\circ) = (32.87)(0.766) = 25.2.$$

By logs:

$$\begin{aligned}
\log 32.87 &= 1.5168 \\
\log \sin 50^\circ &= \underline{\ 9.8843 - 10} \\
\log a &= 11.4011 - 10 \\
a &= 25.18.
\end{aligned}$$

$$\cos A = \frac{b}{c}.$$

Therefore,

$$b = c \cos A.$$

By slide rule:

$$b = (32.87)(\cos 50^\circ) = (32.87)(0.643) = 21.12.$$

By logs:

$$\begin{aligned} \log 32.87 &= 1.5168 \\ \log\cos 50° &= 9.8081 - 10 \\ \hline \log b &= 11.3249 - 10 \\ b &= 21.13. \end{aligned}$$

EXERCISE 2-2

Solve the following right triangles for all unknown parts:

1. $A = 36° 52'$; $c = 5.00$.
2. $B = 15° 16'.8$; $c = 0.0153$.
3. $B = 72° 18' 40''$; $c = 94.832$.
4. $A = 24° 45' 45''$; $c = 1.7384$.
5. $A = 75°$; $c = 0.48725$.
6. $B = 39° 14'$; $c = 4.82$.
7. $B = 61° 18'.6$; $c = 0.0925$.
8. $A = 31° 43' 15''$; $c = 1.9795$.
9. $A = 50° 50'$; $c = 8.073$.
10. $B = 57° 57'$; $c = 52.75$.

C–1. Given two sides (one side and the hypotenuse).

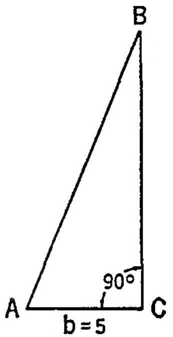


Fig. 2–3.

Example 2–3. In the right triangle, Fig. 2–3, solve for all other parts, given $c = 13$, $b = 5$.

Solution:

By slide rule: $\cos A = \dfrac{b}{c} = \dfrac{5}{13} = 0.3845$

$\angle A = 67° 23'$.

By logs:

$$\begin{aligned} \log 5 &= 10.6990 - 10 \\ \log 13 &= 1.1139 \\ \log\cos A &= 9.5851 - 10 \\ \angle A &= 67° 23'.2. \end{aligned}$$

$$\begin{aligned} \angle B &= 180° - (A + C) = 180° - (90° + 67° 23'.2) \\ &= 22° 36'.8. \end{aligned}$$

By slide rule: $\sin A = \dfrac{a}{c}$.

Therefore, $a = c \sin A = (13)(\sin 67° 23') = (13)(.9231) = 12$.

By logs:

$$\begin{aligned} \log 13 &= 1.1139 \\ \log\sin 67° 23'.2 &= 9.9653 - 10 \\ \hline \log a &= 11.0792 - 10 \\ \log a &= 1.0792 \\ a &= 12. \end{aligned}$$

C–2. Given two sides.

Example 2–4. In the right triangle of Fig. 2–4 solve for all other parts, given $a = 24$, $b = 15$.

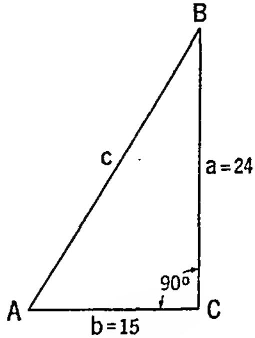


Fig. 2–4.

Solution:

By slide rule: $\tan A = \dfrac{a}{b} = \dfrac{24}{15} = 1.600$

$\angle A = 58°$.

By logs:

$$\begin{aligned} \log 24 &= 1.3802 \\ \log 15 &= 1.1761 \\ \hline \log\tan A &= 0.2041 \\ \angle A &= 57° 59'.7. \end{aligned}$$

$$\angle B = 180° - (A + C)$$
$$= 180° - (57° 59'.7 + 90°) = 32° 0'.3.$$
$$\sin A = \frac{a}{c}.$$

By slide rule: $c = \dfrac{a}{\sin A} = \dfrac{24}{\sin 58°} = \dfrac{24}{0.848} = 28.3.$

By logs:

$$\begin{aligned} \log 24 &= 11.3802 - 10 \\ \log \sin 57° 59'.7 &= 9.9284 - 10 \\ & \qquad\qquad + \\ \log c &= 1.4518 \\ c &= 28.3. \end{aligned}$$

EXERCISE 2–3

Solve the following right triangles for all unknown parts:

1. $b = 1.583$; $c = 7.427$.
2. $b = 14.49$; $a = 19.32$.
3. $a = 0.75861$; $b = 0.48305$.
4. $c = 0.8753$; $a = 0.3261$.
5. $a = 63.2$; $b = 47.4$.
6. $c = 0.815$; $a = 0.489$.
7. $a = 76.854$; $b = 102.472$.
8. $b = 0.260$; $c = 0.335$.
9. $c = 85.62$; $b = 50$.
10. $c = 2.437$; $a = 1.486$.

2. The right-triangle formula. The solutions for Examples 2–1, 2–2, 2–3, and 2–4 have used the trigonometric functions either for slide rule computation or for logarithmic computation. However, there is another method of attack, and this method might well be used as a check upon our results from the trigonometric solution.

From our geometry we know that $c^2 = a^2 + b^2$, and given any two of these values, the third may be found by solving this equation. However, it does not lend itself readily to slide-rule calculation and is therefore quite cumbersome. Several methods, however, have been devised for making it applicable to slide-rule work, and a couple of the simpler ones are worth noting.

Suppose we have given the hypotenuse, c, and one of the other sides such as b, and wish to solve for a. Then, transposing the original equation so as to solve for a, we have

$$a = \sqrt{c^2 - b^2}.$$

Since we have the difference of two squares under the radical sign, we can factor the expression by algebra:

$$a = \sqrt{(c + b)(c - b)}.$$

Now all that is necessary is to find the sum and difference of the quantities given and perform the remainder of the operation directly on the slide rule. We shall check Example 2–3 by this method.

Solution: Example 2–3.

Given $\quad c = 13, \quad b = 5.$

$$a = \sqrt{c^2 - b^2} = \sqrt{(c + b)(c - b)},$$
$$a = \sqrt{(13 + 5)(13 - 5)} = \sqrt{(18)(8)} = \sqrt{144} = 12.$$

The problem is somewhat more complicated when the two legs are given and it is desired to find the hypotenuse. In this case, we have the equation in its original form, $c = \sqrt{a^2 + b^2}$, and the expression under the radical is not factorable. However, it may be put in a form readily adapted to the slide rule.

Multiplying and dividing the right-hand member by b, we get

$$c = b\,\frac{\sqrt{a^2 + b^2}}{b} = b\sqrt{\frac{a^2 + b^2}{b^2}} = b\sqrt{\frac{a^2}{b^2} + \frac{b^2}{b^2}}.$$

Then $$c = b\sqrt{\frac{a^2}{b^2} + 1}.$$

This last expression is now in a form to work out on the slide rule and we shall check Example 2–4 by this method.

Solution: Example 2–4.

$$c = 15\sqrt{\left(\frac{24}{15}\right)^2 + 1} = 28.3.$$

The solution is made easier if b is chosen as the smaller number.

Now set the 15 of the C scale over 24 on the D scale, and the A scale opposite the index of the C scale will read $\left(\frac{24}{15}\right)^2 = 2.56$. Adding 1 to this gives 3.56 on the A scale. Directly underneath this on the D scale will be found its square root (1.888). Set the 1 of the C scale over this square root, move the hairline along to 15 on the C scale, and the answer will be found underneath on the D scale.

In summarizing, then, we may point out that there are two broad methods of solving a right triangle:

(1) By use of the right-triangle formula

$$c^2 = a^2 + b^2.$$

(2) By use of the formulas of the trigonometric functions.

The right-triangle formula is best used where two sides are given (two legs or one leg and the hypotenuse) and the values for these sides are round numbers easily workable.

For all other conditions it is better to use the trigonometric functions by selecting and using the one that contains the desired unknown and no other unknowns. This means the use of the sine, cosine, or tangent.

If a check on the solution is desired, the figure might be drawn approximately to scale or the method not used in the original solution might be employed.

EXERCISE 2–4

1. Check the solutions for Problems 1, 2, and 3 of Exercise 2–3 by the methods given in Article 2.

2. Check the solutions for Problems 4, 5, 6, and 7 of Exercise 2–3.

3. Check the solutions for Problems 8, 9, 10 of Exercise 2–3.

3. Accuracy of results. Certain errors are bound to creep into most results of problems in applied mathematics involving computations. These errors are of two sorts: errors due to inaccuracies in the given data and errors due to limitations in the tables used. Of course, it is not possible to eliminate such errors, but it is possible to get results that recognize such errors and that are sufficiently accurate for the purposes of the problem at hand.

In general, it may be said that a result can be no more accurate than the given data. Therefore, *the number of significant digits in a result should be rounded off so that there will be no more digits than in the least accurate data,* subject, of course, to a practical interpretation. Thus, if two sides of a right triangle are given as 14.49 and 19.32, then the hypotenuse may be computed to be 24.15. This gives the answer to the nearest hundredth of a foot as is given in the data and is therefore correct, whereas another digit in the answer would assume more accuracy than the given data.

In triangles, it is accepted practice to assume that *four-place accuracy in the sides generally gives accuracy in the angle to the nearest minute, and five-place accuracy in the sides gives accuracy in the angles to the nearest tenth of a minute.*

In the use of tables of logarithms or functions, it is quite generally agreed that *four-place accuracy* will result from the use of *four-place tables,* whereas *five-place accuracy* will result from the use of *five-place tables.* Four-place accuracy is generally acceptable, and slide-rule computation comes within reasonable limits of this, although the choice depends upon a study of the instruments used in taking the measurements.

4. Vectors and components. In technical work a great many problems are attacked mathematically by treating quantities like forces as *vectors.* Therefore, the technical student very early in his study of mathematics should get acquainted with vectors and the solution of problems involving them.

A vector is used to represent a quantity that has both magnitude and direction. Thus, if a force of 100 lb acts in a certain direction upon a body, the force may be represented by a vector drawn to a definite scale and in the direction in which the force is acting. For instance, in Fig. 2–5, with a scale of 1 in. = 50 lb, the line OA drawn 2 in. long and at an angle of 30° with the horizontal represents a force of 100 lb acting at an angle of 30° on the body, and is termed the vector of that force.

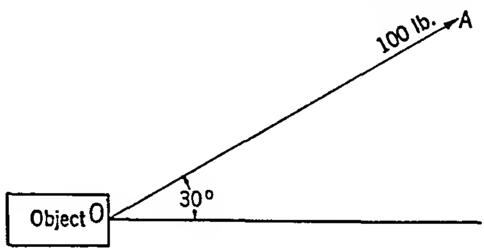


Fig. 2–5.

It is quite possible that more than one force may be acting upon an object, and it is this condition that gives rise to the necessity of mathe-

matical treatment. In Fig. 2–6 are shown two forces of 50 lb and 75 lb, respectively, acting upon an object at the angles shown.

There is a law in physics which states that two forces acting upon an object at a single point have the same effect as the resultant or sum of the two. Therefore, we can draw the two forces of Fig. 2–6 from the origin of a rectangular coördinate system and find their resultant by getting the diagonal of the parallelogram they form.

In Fig. 2–7, the two forces OA and OB of Fig. 2–6 are drawn to scale and at their proper angles. Then by completing the parallelogram, the diagonal OC represents the sum or resultant of the forces OA and OB.

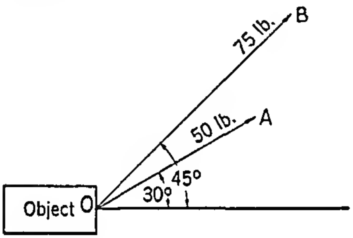


Fig. 2–6.

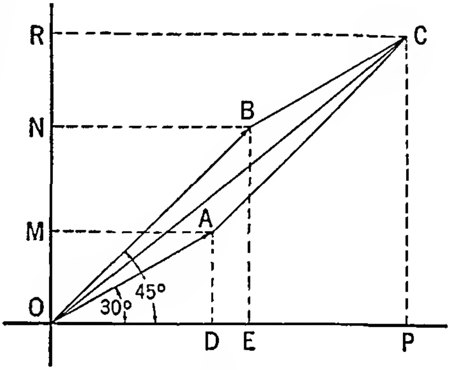


Fig. 2–7.

One method of treating such vectors in order to find the resultant is by using their projections upon the x-axis and the y-axis. In Fig. 2–7 the projection upon the x-axis of the force OA is the distance OD, found by dropping a perpendicular from the end of OA to the x-axis. This projection is called the *horizontal component* of force OA. Likewise the projection upon the y-axis of the force OA is the distance OM, found by drawing a perpendicular from the end of OA to the y-axis. This projection is called the *vertical component* of OA. In a similar manner, the horizontal components of force OB and resultant OC are OE and OP, respectively, and the vertical components of force OB and resultant OC are ON and OR, respectively.

It is evident, from an inspection of Fig. 2–7, that the horizontal components OE and OD can be added directly, since they lie in the same straight line, and the same statement can be made for the vertical components OM and ON. And these additions give the components of the resultant:

$$OE + OD = OP,$$
$$OM + ON = OR.$$

Furthermore, since each of the triangles OAD, OBE, and OCP is a right triangle, the right-triangle formula holds true for each one.

$$(OD)^2 + (DA)^2 = (OA)^2;$$
$$(OE)^2 + (EB)^2 = (OB)^2;$$
$$(OP)^2 + (PC)^2 = (OC)^2.$$

Therefore, it follows that

$$(OE + OD)^2 + (OM + ON)^2 = (OC)^2.$$

It must be remembered that the horizontal components of any force in the second or third quadrants will be negative, and the vertical component of any force in the third or fourth quadrant will be negative. Therefore, *the numerical sign of the components must be taken into consideration in any additions to find components of a resultant.*

EXAMPLE 2–5. Find the components of the forces given in Fig. 2–6. Also, find the resultant and the angle it makes with the x-axis.

Solution:

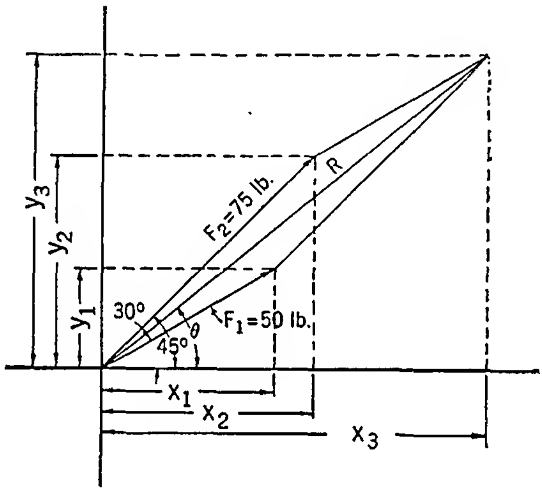


Fig. 2–8.

Let horizontal component of force $F_1 = X_1$
and horizontal component of force $F_2 = X_2$.
Let vertical component of force $F_1 = Y_1$
and vertical component of force $F_2 = Y_2$.
Let horizontal component of resultant $R = X_3$
and vertical component of resultant $R = Y_3$.

$$\begin{aligned}
\text{Now } X_1 &= F_1 \cos 30° = 50 \cos 30° = (50)(0.866) = 43.3 \text{ lb},\\
Y_1 &= F_1 \sin 30° = 50 \sin 30° = (50)(0.5) = 25 \text{ lb},\\
X_2 &= F_2 \cos 45° = 75 \cos 45° = (75)(0.707) = 53.03 \text{ lb},\\
Y_2 &= F_2 \sin 45° = 75 \sin 45° = (75)(0.707) = 53.03 \text{ lb}.\\
\text{Now } X_3 &= X_1 + X_2 = 43.3 + 53.03 = 96.33 \text{ lb},\\
Y_3 &= Y_1 + Y_2 = 25 + 53.03 = 78.03 \text{ lb},\\
R &= \sqrt{(X_3)^2 + (Y_3)^2} = \sqrt{(96.33)^2 + (78.03)^2} = 124 \text{ lb}.\\
\tan\theta &= \frac{Y_3}{X_3} = \frac{78.03}{96.33} = 0.81,\\
\theta &= 39°.
\end{aligned}$$

An inspection of this solution will make it plain that if the resultant R is given along with one of the sides such as F_1, then the procedure would be to subtract the horizontal and vertical components of F_1 from those of R and then proceed in the same way to find F_2.

An application of the above considerations in the electrical field is in the use of vectors to represent voltages and currents in alternating-current systems. Since an alternating-current voltage or current follows the variations of the sine function as it goes from 0° to 360°, it may be treated as a rotating vector, which condition makes it possible to use the same kind of solution as was used in Example 2–5.

EXAMPLE 2–6. Two alternating-current voltages, $V_1 = 100$ and $V_2 = 120$, form an angle of 30° with each other. Find the resultant and the angle it makes with one of the others.

Solution:

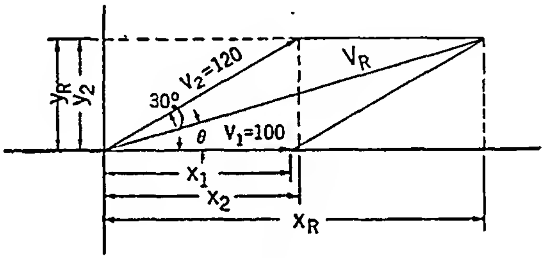


Fig. 2–9.

Draw V_1 along the x-axis, and 30° from this construct V_2. It is best to construct our vectors for voltages or currents in a counterclockwise direction, since vector rotation in alternating-current systems is assumed this way.

Since V_1 has been placed along the x-axis, its vertical component is zero, and its horizontal component is equal to V_1.

$$
\begin{aligned}
X_1 &= V_1 = 100\\
Y_1 &= V_1 \sin 0° = (100)(0) = 0\\
X_2 &= V_2 \cos 30° = (120)(0.866) = 103.9\\
Y_2 &= V_2 \sin 30° = (120)(0.5) = 60\\
X_R &= X_1 + X_2 = 100 + 103.9 = 203.9\\
Y_R &= Y_1 + Y_2 = 0 + 60 = 60\\
V_R &= \sqrt{(X_R)^2 + (Y_R)^2} = \sqrt{(203.9)^2 + (60)^2} = 212.5.\\
\cos\theta &= \frac{X_R}{V_R} = \frac{203.9}{212.5} = 0.9595\\
\theta &= 16° \, 22'.
\end{aligned}
$$

(*Note:* The symbols X and Y have been used here to represent the horizontal and vertical components, respectively, but in many electrical texts these components are represented by the symbols h and v.)

5. Vectors in complex notation. In Chapter 3, Part I, we have studied complex numbers and the methods of treating them. Now we can make good use of this notation in our solutions for problems involving vectors.

Since a complex number consists of a real number and an imaginary number at right angles to each other, it is evident that the real number is along the horizontal axis and the imaginary number is along or parallel to the vertical axis. Thus, the complex number, $3 + j4$, is made up of two parts; one part, 3, is on the horizontal axis, and the other part, $+j4$, is parallel to the vertical axis. Therefore, the complex number represents a right triangle with the real number part as one side of the triangle on the horizontal axis and the imaginary number part as the other side parallel to the vertical axis. The hypotenuse of the right triangle can

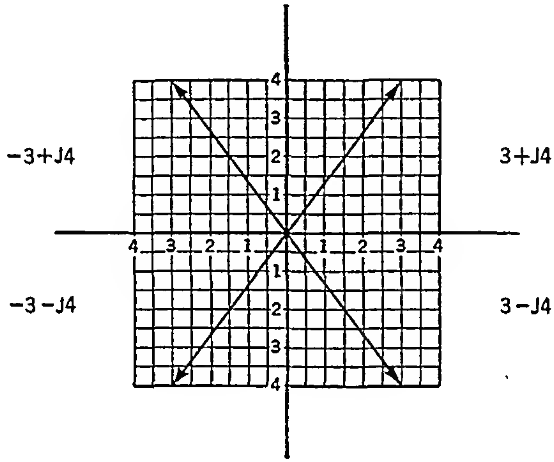


Fig. 2–10.

be determined readily by extracting the square root of the sum of the squares of the two sides as in any right triangle. Thus, from the complex number, $3 + j4$, representing a triangle with sides of 3 and 4, we find the hypotenuse to be $\sqrt{3^2 + 4^2}$ or 5. See Figure 2–10.

From the foregoing, it is clear that the real part of the complex number is the same as a horizontal component and the imaginary part is the same as a vertical component. This, then, presents us with another method of defining vector quantities in terms of their horizontal and vertical components. *The vector is written as a complex number with the horizontal component as the first term and the vertical component as the second term.* The quadrant in which the vector lies is determined by the signs preceding the component. If both signs are positive, the vector is in the first quadrant; if the first term is negative and the second term positive, the vector is in the second quadrant; if both terms are negative, the vector is in the third quadrant; and if the first term is positive and the second term negative, the vector is in the fourth quadrant. Thus a

vector of length 5, a horizontal component of 3, and a vertical component of 4 is written as: $3 + j4$ for a first quadrant position; $-3 + j4$ for a second quadrant position; $-3 - j4$ for a third quadrant position; and $+3 - j4$ for a fourth quadrant position.

Since the horizontal component is found by multiplying a vector by the cosine of the angle it makes with the x-axis and the vertical component is found by multiplying the vector by the sine of the same angle, the vector may be written in the form

$$V(\cos \theta + j \sin \theta)$$

where V is the value of the vector and θ is the angle measured from the x-axis to the vector in a counter-clockwise direction. Then, for the vector 5, that we have been discussing, $\cos \theta = 3/5$ and $\sin \theta = 4/5$. Therefore, we have

$$5(\cos \theta + j \sin \theta) = 5(0.6 + j0.8) = 3 + j4.$$

Vectors written in the complex number form are called *rectangular vectors* and may be added, subtracted, multiplied, or divided in the same manner as any complex numbers.

EXAMPLE 2–7. Solve Example 2–5 by the complex number notation.

Solution: Let V_1 = vector 50 at its angle of 30°; V_2 = vector 75 at its angle of 45°; and V_T = the resultant vector.

$$\begin{aligned} V_1 &= 50(\cos 30° + j \sin 30°) \\ &= 50(0.866 + j0.5) \\ &= 43.3 + j25. \\ V_2 &= 75(\cos 45° + j \sin 45°) \\ &= 75(0.707 + j0.707) \\ &= 53.03 + j53.03. \end{aligned}$$

Therefore, $V_T = V_1 + V_2 = 43.3 + j25 + 53.03 + j53.03.$

Or $\quad V_T = 96.33 + j78.03.$

Then $\quad V_T = \sqrt{(96.33)^2 + (78.03)^2} = 124 \text{ lb}.$

The angle that V_T makes with the x-axis can be found by dividing the total vertical value by the total horizontal value to get the tangent.

$$\operatorname{Tan} \theta = \frac{V_1 \sin 30° + V_2 \sin 45°}{V_1 \cos 30° + V_2 \cos 45°}.$$

$$\operatorname{Tan} \theta = \frac{25 + 53.03}{43.3 + 53.03} = \frac{78.03}{96.33} = 0.81.$$

$$\theta = 39°$$

EXAMPLE 2–8. Solve Example 2–6 by the complex number notation.

Solution: $\quad V_1 = 100(\cos 0° + j \sin 0°),$

$\quad V_2 = 120(\cos 30° + j \sin 30°).$

$$\begin{aligned}
&\text{Then} & V_1 &= 100(1 + j0) = 100,\\
&\text{and} & V_2 &= 120(.866 + j0.5) = 103.9 + j60.\\
&\text{Therefore} & V_R &= 100 + 103.9 + j60 = 203.9 + j60,\\
& & V_R &= \sqrt{(203.9)^2 + (60)^2} = 212.5.\\
& & \text{Tan } \theta &= \frac{100 \sin 0^\circ + 120 \sin 30^\circ}{100 \cos 0^\circ + 120 \cos 30^\circ} = \frac{60}{100 + 103.9}\\
& & &= \frac{60}{203.9} = 0.2939.\\
& & \theta &= 16^\circ\ 22'.
\end{aligned}$$

The angle might also have been determined from the cosine.

$$\begin{aligned}
\text{Cos } \theta &= \frac{203.9}{212.5} = 0.9595,\\
\theta &= 16^\circ\ 22'.
\end{aligned}$$

6. Polar vector notation. A vector may be defined also in terms of its magnitude and the angle it makes with the positive x-axis. When defined thus, it is termed a *polar vector.* The magnitude of the vector is called the *modulus* and its direction angle is called the *argument.* A vector of magnitude 50 and an angle of 30° would be written $50\underline{/30^\circ}$. The sign, $\underline{/\ }$, indicates counter clockwise or positive rotation. Clockwise or negative rotation would be indicated by the reversed sign, $\overline{/\ }$.

Examples:

$$\begin{aligned}
&50\underline{/30^\circ} && \text{first quadrant,}\\
&50\underline{/120^\circ} && \text{second quadrant,}\\
&50\overline{/120^\circ} && \text{third quadrant,}\\
&50\overline{/30^\circ} && \text{fourth quadrant.}
\end{aligned}$$

In Example 2–7, we would write:

$$\begin{aligned}
V_1 &= 50\underline{/30^\circ} = 50\ (\cos 30^\circ + j \sin 30^\circ) = 43.3 + j25,\\
V_2 &= 75\underline{/45^\circ} = 75\ (\cos 45^\circ + j \sin 45^\circ) = 53.03 + j53.03,\\
V_T &= 124\underline{/39^\circ} = 124\ (\cos 39^\circ + j \sin 39^\circ) = 96.33 + j78.03.
\end{aligned}$$

In Example 2–8:

$$\begin{aligned}
V_1 &= 100\underline{/0^\circ} = 100\ (\cos 0^\circ + j \sin 0^\circ) = 100,\\
V_2 &= 120\underline{/30^\circ} = 120\ (\cos 30^\circ + j \sin 30^\circ) = 103.9 + j60,\\
V_R &= 212.5\underline{/16^\circ\ 22'} = 212.5\ (\cos 16^\circ\ 22' + j \sin 16^\circ\ 22') = 203.9 + j60.
\end{aligned}$$

Vectors may be multiplied or divided when expressed in the polar form, but for addition or subtraction must be changed to the rectangular form.

The product of two polar vectors is equal to the product of their moduli and the sum of their arguments. The product of two rectangular vectors is determined from the principles of algebra as shown in Chapter 3, Part 1.

The quotient of two polar vectors is equal to the quotient of their moduli and the difference of their arguments. The quotient of two rectangular vectors is determined from the principles of algebra as shown in Chapter 3, Part 1.

A polar vector raised to the nth power is equal to the nth power of the modulus and the product of n and the argument.

Thus: $(E\underline{/\theta})^n = E^n\underline{/n\theta}$.

EXAMPLE 2–9. Multiply $5\underline{/30^\circ}$ by $4\overline{/20^\circ}$.

Solution:

$$(5\underline{/30^\circ})(4\overline{/20^\circ}) = (5)(4)\underline{/30^\circ + (-20^\circ)}$$
$$= 20\underline{/10^\circ}.$$

EXAMPLE 2–10. Divide $250\underline{/120^\circ}$ by $50\underline{/70^\circ}$.

Solution:

$$\frac{250\underline{/120^\circ}}{50\underline{/70^\circ}} = \frac{250}{50}\underline{/120^\circ - 70^\circ} = 5\underline{/50^\circ}.$$

EXAMPLE 2–11. Evaluate each of the following:

$$8\underline{/45^\circ} + 10\underline{/30^\circ},$$
$$22\underline{/60^\circ} - 12\underline{/45^\circ}.$$

Solution:

$$8\underline{/45^\circ} = 8(\cos 45^\circ + j \sin 45^\circ) = 8(0.707 + j0.707) = 5.656 + j5.656,$$
$$10\underline{/30^\circ} = 10(\cos 30^\circ + j \sin 30^\circ) = 10(0.866 + j0.5) = 8.66 + j5.$$

Therefore, $8\underline{/45^\circ} + 10\underline{/30^\circ} = 5.656 + j5.656 + 8.66 + j5 = 14.316 + j10.656$.

Changing the result back to polar form,

$$\sqrt{(14.316)^2 + (10.656)^2} = 17.85,$$
$$\tan\theta = \frac{10.656}{14.316} = 0.745,$$

or

$$\theta_1 = 36^\circ\ 41' \text{ (approx.)}.$$

Then $8\underline{/45^\circ} + 10\underline{/30^\circ} = 17.85\underline{/36^\circ\ 41'}$,

$$22\underline{/60^\circ} = 22(\cos 60^\circ + j \sin 60^\circ) = 22(0.5 + j0.866) = 11 + j19.05,$$
$$12\underline{/45^\circ} = 12(\cos 45^\circ + j \sin 45^\circ) = 12(0.707 + j0.707) = 8.48 + j8.48.$$

Therefore, $22\underline{/60^\circ} - 12\underline{/45^\circ} = 11 + j19.05 - (8.48 + j8.48) = 2.52 + j10.57$.

Changing this result back to polar form:

$$\sqrt{(2.52)^2 + (10.57)^2} = 11.2,$$
$$\tan\theta_2 = \frac{10.52}{2.52} = 4.315,$$

or $$\theta_2 = 76^\circ\ 57' \text{ (approx.)}.$$

Then $22/\underline{60^\circ} - 12/\underline{45^\circ} = 11.2/\underline{76^\circ\ 57'}$.

EXAMPLE 2–12. Determine the value of

$$(9/\underline{30^\circ})^2;\ (16/\underline{120^\circ})^{1/2}.$$

Solution:

$$(9/\underline{30^\circ})^2 = 9^2/\underline{(2)(30^\circ)} = 81/\underline{60^\circ},$$
$$(16/\underline{80^\circ})^{1/2} = 16^{1/2}/\underline{(1/2)(80^\circ)} = 4/\underline{40^\circ}.$$

EXERCISE 2–5

Write the following rectangular vectors in polar vector form:

1. $6 + j8$.
2. $8 - j3$.
3. $-7 + j5$.
4. $-9 - j4$.
5. $10 - j7$.
6. $-12 - j11$.
7. $-11 + j15$.
8. $30 + j20$.
9. $21 - j13$.
10. $-14 + j12$.

Write the following polar vectors in rectangular vector form:

11. $10/\underline{30^\circ}$.
12. $12/\underline{20^\circ}$.
13. $15.5/\underline{60.3^\circ}$.
14. $18.6/\underline{50.6^\circ}$.
15. $13.4/\underline{45.8^\circ}$.
16. $17.5/\underline{22.5^\circ}$.
17. $19.3/\underline{13.5^\circ}$.
18. $16.2/\underline{75^\circ}$.
19. $25.3/\underline{45^\circ}$.
20. $21.7/\underline{50.2^\circ}$.

Perform the indicated operations:

21. $(5.2/\underline{15^\circ})(3.7/\underline{25^\circ})$.
22. $(4.9/\underline{35^\circ})(7.6/\underline{22^\circ})$.
23. $(10.6/\underline{65.5^\circ})(12.8/\underline{35.2^\circ})$.
24. $(11.3/\underline{12.3^\circ})(13.4/\underline{33.4^\circ})$.
25. $(21.8/\underline{42.6^\circ})(32.6/\underline{18.5^\circ})$.
26. $\dfrac{25.2/\underline{40^\circ}}{8.4/\underline{22^\circ}}$.
27. $\dfrac{14.3/\underline{27^\circ}}{35.8/\underline{12.2^\circ}}$.
28. $\dfrac{17.9/\underline{42^\circ}}{52.5/\underline{11.7^\circ}}$.
29. $\dfrac{43.4/\underline{14.5^\circ}}{11.6/\underline{36^\circ}}$.
30. $\dfrac{6.35/\underline{51.6^\circ}}{4.68/\underline{38.7^\circ}}$.
31. $7.25/\underline{25^\circ} + 3.82/\underline{35^\circ}$.
32. $12.9/\underline{42^\circ} + 11.2/\underline{20^\circ}$.
33. $15.6/\underline{18.6^\circ} + 13.8/\underline{22.5^\circ}$.
34. $27.3/\underline{16.4^\circ} + 23.9/\underline{25.8^\circ}$.
35. $33.5/\underline{61.3^\circ} + 43.5/\underline{43.3^\circ}$.
36. $25/\underline{13^\circ} - 17/\underline{10^\circ}$.
37. $32/\underline{35.6^\circ} - 18.5/\underline{20.7^\circ}$.
38. $35.8/\underline{63.8^\circ} - 22.2/\underline{42.6^\circ}$.
39. $53.2/\underline{21.5^\circ} - 28.6/\underline{32.2^\circ}$.
40. $48.4/\underline{42.2^\circ} - 24.6/\underline{28.7^\circ}$.

EXERCISE 2–6

1. Two forces of 80.6 lb and 42.3 lb act upon a body. The first force acts at 15° to the horizontal and the second at 40° to the horizontal. Find the resultant force and the angle it makes with the first one.

2. The two adjacent sides of a parallelogram are equal to 45 and 35, respectively, with an angle between them of 42° 30′. Find the resultant and the angle it makes with one of the sides.

3. Two voltages of 120 and 110 respectively make an angle of 75° 16′ with each other. Find the value of the resultant voltage, and the angle it makes with either of the other voltages.

4. A circuit voltage of 120 is made up of two parts and makes an angle of 15° 36′ with the horizontal axis. If one of the parts is equal to 60 v along the horizontal, what is the value of the other part and what angle does it make with the x-axis?

5. Find the force necessary to hold a 20-lb ball on a plane inclined 15° to the horizontal. *Hint:* Divide the weight into two components, one parallel to the plane and the other perpendicular to the plane. Then solve the triangle formed for the component parallel to the plane.

6. A resultant force of 150 lb acts upon a body at an angle of 39° 39′ to the horizontal. If this resultant is made up of a 75-lb force along the horizontal axis and an unknown, find the unknown force and the angle it makes with the x-axis.

7. Angles of elevation and depression. *The angle of elevation of an object is defined as the angle above a horizontal plane and between a line drawn from the observer to the object and a horizontal line.* Thus in Fig. 2–11 an observer stands at point A and sights at the top of a building. Then the angle of elevation of the top of the building is the angle θ. The distance AB is called the *line of sight.*

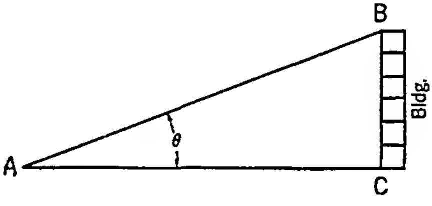


Fig. 2–11.

The angle of depression of an object is defined as the angle below a horizontal plane and between a line drawn from the observer to the object and a horizontal line. Thus, in Fig. 2–12 an observer stands on a hill at point A

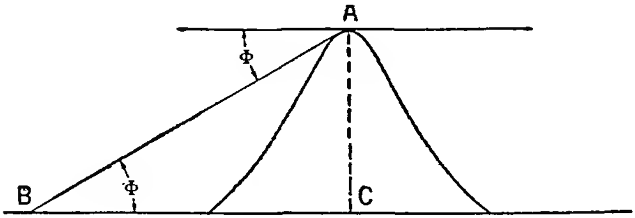


Fig. 2–12.

and sights at an object on the plain below him. Then the angle of depression of the object at B is the angle Φ and the distance AB is called the *line of sight.*

It will be observed from Figs. 2–11 and 2–12 that the relative position of the object with respect to the observer determines whether the angle is one of elevation or one of depression. In Fig. 2–11, the object is above the observer and there results an angle of elevation, while in Fig. 2–12, the object is below the observer and there results an angle of depression.

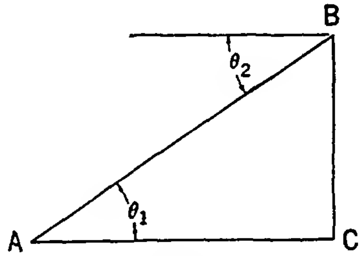


Fig. 2–13.

It should be noted also that the angle of elevation from an observer to an object has exactly the same value as the angle of depression from the object to the observer, because the two angles are alternate interior angles of a line crossing two parallel lines. Such a condition, geometrically, makes the two angles equal, as in Fig. 2–13 where angle θ_1, an angle of elevation, is equal to angle θ_2, an angle of depression.

EXAMPLE 2–13. How tall is a man who casts a shadow 10 ft long when the angle of elevation of the sun is 30°?

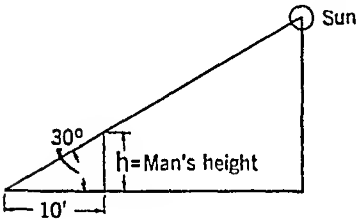


Fig. 2–14.

Solution:

$$\tan 30° = \frac{h}{10}$$

$$h = 10 \tan 30° = (10)\frac{\sqrt{3}}{3}$$

$$= \frac{17.32}{3} = 5.77 \text{ ft} = 5 \text{ ft } 9\frac{1}{4} \text{ in.}$$

8. Angle of inclination of a line. *The angle of inclination of a line is the angle made with the horizontal and does not differ fundamentally from an*

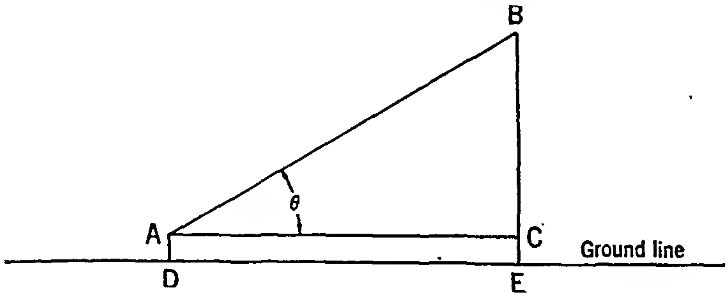


Fig. 2–15.

angle of elevation. The term is used in connection with a line run on sloping ground, and the inclination is positive if the line slopes forward, negative if it slopes backward. Thus in Fig. 2–15, if AC is horizontal and parallel

to the ground line, then the angle θ is the inclination of the line AB and in this case is positive because the line AB slopes forward. The horizontal distance, AC, between the ends of the line AB is called the *course of AB.*

9. Bearing of a line. *The bearing of a line is defined as the angle the line makes with a north and south line.* Therefore, the angle always starts from the north or south line and proceeds toward the east or west. In writing the bearing of a line, the letter N or S is written first, then the angle of deviation from the north or south, and finally E or W denoting the direction toward which the line is rotating. In Fig. 2–16, the bearing of OA is N 30° E and the bearing of OB is S 45° W.

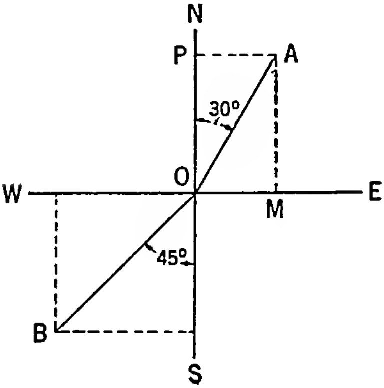


Fig. 2–16.

The distance OP in Fig. 2–16 is called the *latitude* of OA and the distance OM is called the *departure* of OA.

10. Angle subtended by a line. *An angle is said to be subtended by a line when the sides forming the angle meet the end of the line.* Thus, in Fig. 2–17, the chord AB subtends the angle AOB.

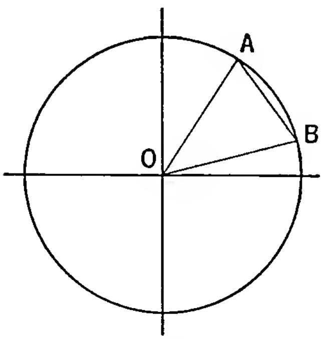


Fig. 2–17.

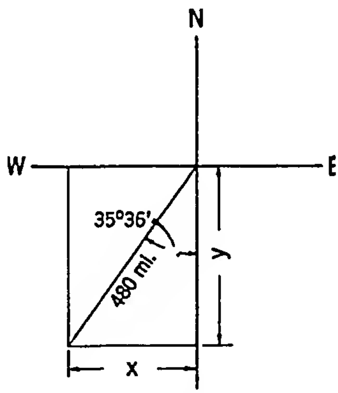


Fig. 2–18.

Example 2–14. A ship's bearing is S 35° 36′ W. How far south of its original starting position will it be at the end of 24 hr if its average speed is 20 mph? How far west?

Solution:

Total distance traveled = (20)(24) = 480 mi; $\sin 35°\,36' = \dfrac{x}{480}$.

$x = (480) \sin 35° 36' = (480)(0.5821) = 279.4$ mi; $\cos 35° 36' = \frac{y}{480}$.

$y = 480 \cos 35° 36' = (480)(0.8131) = 390.3$ mi.

This may be checked by the right-triangle formula method discussed in Article 2.

EXERCISE 2–7

1. From a point on the ground 160 ft from the foot of a flagpole, the angle of elevation is found to be 34° 16′. What is the height of the flagpole?

2. From the top of a cliff the angle of depression of a boat 300 ft from the base of the cliff is found to be 25° 18′. What is the height of the cliff in feet?

3. From a point 250 ft from the base of a building, the angle of elevation of the top of a flagpole on the building is found to be 53° 17′ and the angle of elevation of the top of the building is found to be 43° 43′. Find the heights of the building and the flagpole.

4. From the top of a building the angles of depression of two road markers 126.13 ft apart are found to be 12° and 24° respectively. What is the height of the building? (Two simple simultaneous equations are required to solve.)

5. A ship traveling at the rate of 18 mph has made 350 miles in a northward direction and 250 miles in an eastward direction. What is its bearing and how far has it traveled?

6. A ship travels 260 mi in a direction N 31° 45′ E and then changes its bearing to travel 210 miles N 68° 28′ E. If it had traveled in a direct line to its destination, how far would it have sailed and what would have been its bearing? Use the component method.

11. The solution of the oblique triangle by right-triangle methods. In oblique triangles, the same notation is used as for right triangles to represent the sides and angles, but the letter C does not now indicate a right angle. A general method of procedure in solving oblique triangles consists of dividing the triangle into two right triangles by constructing a perpendicular from a vertex to the opposite side and then solving these right triangles.

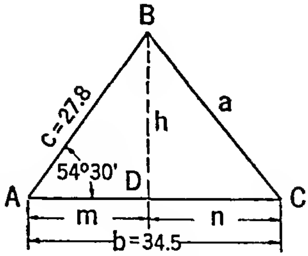


Fig. 2–19.

EXAMPLE 2–15. Solve the oblique triangle ABC when $A = 54° 30'$; $b = 34.5$; $c = 27.8$.

Solution:

Erect a perpendicular from vertex B to side b, dividing the original triangle into two right triangles ABD and BCD, and label this perpendicular h, as shown in Fig. 2–19.

From the figure,

$$h = c \sin 54° 30'$$
$$h = (27.8)(0.8141) = 22.63;$$
$$m = c \cos 54° 30'$$
$$m = (27.8)(0.5807) = 16.14;$$
$$n = b - m = 34.5 - 16.14 = 18.36;$$
$$\tan C = \frac{h}{n} = \frac{22.63}{18.36} = 1.231$$
$$C = 50° 54'.$$
$$B = 180° - (A + C) = 180° - (54° 30' + 50° 54') = 74° 36'.$$
$$a = \frac{n}{\cos C} = \frac{18.36}{\cos 50° 54'} = \frac{18.36}{0.6307} = 29.1.$$

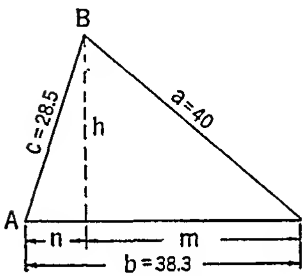


Fig. 2–20.

EXAMPLE 2–16. Solve the oblique triangle ABC when $a = 40$; $c = 28.5$; $b = 38.3$.

Solution:
$$h^2 = c^2 - n^2$$
$$h^2 = a^2 - m^2.$$

Therefore,

$$c^2 - n^2 = a^2 - m^2$$
$$m^2 - n^2 = a^2 - c^2.$$

Factoring:

$$(m + n)(m - n) = (a + c)(a - c).$$

Thus
$$m - n = \frac{(a + c)(a - c)}{m + n} = \frac{(a + c)(a - c)}{b},$$
$$m - n = \frac{(40 + 28.5)(40 - 28.5)}{38.3}.$$
$$m - n = \frac{(68.5)(11.5)}{38.3} = 20.6.$$

Solving simultaneously,

$$\begin{array}{rl} m - n &= 20.6 \\ m + n &= 38.3 \\ \hline 2m &= 58.9 \end{array}$$
$$m = 29.45;$$
$$n = b - m = 38.3 - 29.45 = 8.85;$$
$$\cos A = \frac{n}{c} = \frac{8.85}{28.5} = 0.3105$$
$$A = 71°55'.$$
$$\cos C = \frac{m}{a} = \frac{29.45}{40} = 0.7363$$
$$C = 42° 35'.2.$$
$$B = 180° - (A + C)$$
$$= 180° - (71° 55' + 42° 35'.2) = 65° 28'.8.$$

It should be noted from these two cases that *the largest angle is opposite the largest side* and *the smallest angle is opposite the smallest side*, which gives a means of drawing the triangle to approximate proportions from the original data. However, it must not be inferred that the angles change in the same ratio as the sides change, because this is not true.

Since from our geometry we know that the area of any triangle is equal to one half the product of the base and altitude, it is an easy matter to get the area of any triangle. It must be remembered, however, that the altitude must be taken to the side used as a base.

Thus in Example 2–15 if we denote the area of the triangle ABC by K, then

$$K = \frac{1}{2}bh.$$

Therefore, since $b = 34.5$ and $h = 22.63$,

$$K = \left(\frac{1}{2}\right)(34.5)(22.63) = (17.25)(22.63) = 390.4.$$

In Example 2–16 the area of triangle ABC is

$$K = \frac{1}{2}bh = \left(\frac{1}{2}\right)(38.3)(h).$$

Now $\qquad h = c \sin A = (28.5)(\sin 71^\circ\ 55') = 27.09.$

Therefore, $\qquad K = \left(\frac{1}{2}\right)(38.3)(27.09) = 518.8.$

EXERCISE 2–8

Solve the following oblique triangles for all unknown parts and find the area of each one:

1. $B = 41^\circ\ 27'.8$; $a = 0.4372$; $c = 0.6854$.
2. $A = 25^\circ\ 16'$; $b = 6.731$; $c = 9.438$.
3. $a = 21.43$; $b = 35.62$; $c = 28.436$.
4. $a = 61.83$; $c = 74.53$; $b = 93.26$.
5. $A = 34^\circ\ 20'$; $C = 75^\circ\ 16'$; $a = 115.6$.
6. $B = 63^\circ\ 47'$; $C = 70^\circ\ 10'$; $c = 0.218$.

12. Applied problems. We have discussed a few types of applied problems in this chapter so far but it may be well also to take up some cases of other typical problems in order to gain familiarity with methods of attack.

EXAMPLE 2–17. It is desired to find the width of a river. From a point on one bank the angle of elevation of the top of a tower directly opposite on the other bank is measured as $16^\circ\ 47'.2$. The tower is known to be 200 ft high. What is the width of the river?

Solution:

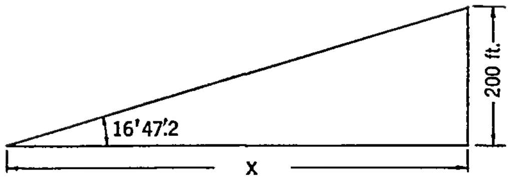


Fig. 2–21.

Let width of river $= x$.

Then $\tan 16° 47'.2 = \frac{200}{x}$,

or $$x = \frac{200}{\tan 16° 47'.2} = \frac{200}{0.3017} = 663 \text{ ft.}$$

EXAMPLE 2–18. Find the length of an open belt connecting two pulleys whose radii are 5 in. and 3 in., respectively. The distance between the centers of the pulleys is 24 in.

Solution:

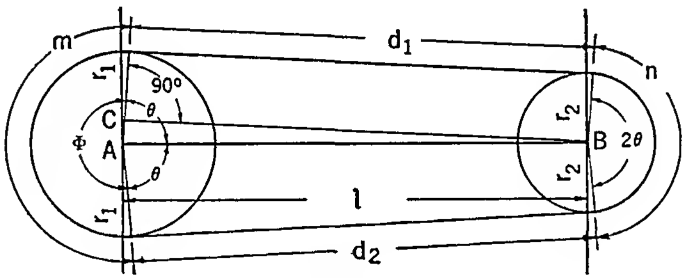


Fig. 2–22.

Given
$$r_1 = 5 \text{ in.},$$
$$r_2 = 3 \text{ in.},$$
$$l = 24 \text{ in.}$$

To find length of belt $= m + n + d_1 + d_2$

Draw BC parallel to d_1, making a right angle at C
In the right triangle ABC

$$\cos \theta = \frac{2}{24} = \frac{1.0}{12} = 0.0833,$$
$$\theta = 85° 13',$$
$$2\theta = 170° 26' = 2.97 \text{ radians},$$
$$\Phi = 360° - 170° 26' = 189° 34' = 3.31 \text{ radians},$$
$$m = (r_1)(\Phi) = (5)(3.31) = 16.55 \text{ in.},$$
$$n = (r_2)(2\theta) = (3)(2.97) = 8.91 \text{ in.},$$
$$d_1 = d_2 = l \sin \theta = (24) \sin 85° 13' = 23.92 \text{ in.}$$

Then length of belt $= m + n + d_1 + d_2$
$= 16.55$ in. $+ 8.91$ in. $+ 23.92$ in. $+ 23.93$ in.
$= 73.3$ in.

EXERCISE 2-9

1. A 40-ft ladder resting against the side of a barn reaches a point 24 ft from the ground. What angle does the ladder make with the ground?

2. The angle of elevation of the top of a flagstaff is found to be 35° 16′ when measured from a point 200 ft from its base. How high is the flagstaff?

3. Two pulleys have radii of 6 in. and 4 in., respectively. If the pulleys are 12 ft apart, how long will an open belt be that connects them?

4. How long a crossed belt is necessary in Problem 3?

5. The Great Pyramid casts a shadow 915 ft long when the angle of elevation of the sun is 27° 37′.9. What is the height of the pyramid?

6. If a motor voltage of 110 makes an angle of 43° 15′ with the horizontal axis and a resultant voltage of 150 makes an angle of 33° 25′ with the horizontal axis, what will be the value of voltage across the piece of equipment that must be put in series with the motor so that it may be used on 150 v? What angle does this latter voltage make with the horizontal?

7. The bearing of a ship sailing at the rate of 22 mph is S 65° 17′.2 W. At what rate is the ship sailing southward?

8. A body is acted upon by two forces of 526.8 lb and 371.2 lb, respectively. The bearing of the first is N 50° 15′ E and of the second is S 23° 24′ E. Find the size and bearing of the resultant.

9. A certain force just balances the combined effect of a force of 213 lb pulling N 72° 48′ E and another of 175 lb pulling N 50° 39′ E. Find its value and direction.

10. A flagpole is situated on top of a building 45 ft high. The angles of elevation of the top and bottom of the flagpole from a point on the ground are 45° and 30° respectively. Find the height of the flagpole and the distance from the observer to the base of the building.

11. Show that in any triangle

$$\sin \frac{A + B}{2} = \cos \frac{C}{2}.$$

12. The Washington Monument is 555 ft high. From a certain point on the ground the angle of elevation of the top is 75° 6′. How far back in the same vertical plane must an observer move to make the angle of elevation 37° 35′.2?

13. Two sides of a parallelogram are 5 in. and 7 in. and the angle included between them is 64° 15. Find the long diagonal and the area.

voltage. Find the unknown voltage and the angle it makes with the horizontal.

45. Two forces of 107.3 and 218.4 act at right angles to each other. Determine their resultant and the angle it makes with each of the forces.

46. The adjacent sides of a lot in the shape of a parallelogram are equal to 201.65 ft and 118.34 ft. The angle between them is 46° 18′.4. Find the two diagonals.

47. Find the length of an open belt connecting two pulleys whose radii are 1.5 ft and 3.5 ft. The distance between the centers of the pulleys is 12 ft.

48. Find the length of the crossed belt necessary for Problem 47.

49. A tower which is 257.8 ft high casts a shadow 324.2 ft long. What is the angle of elevation of the sun?

50. The sides of a rectangular box are 45.27 in.; 38.63 in. and 24.2 in. What is the length of the body diagonal?

51. A pendulum is 30 in. long and the horizontal distance between the ends of its swing is 7.5 in. Determine the angle through which it swings and the distance traveled by the end from one extreme position to the other.

52. In a horizontal distance of 100 ft, a hill rises 13.52 ft. Determine the difference in elevation between two points 350.7 ft apart on the hillside.

53. The angle of elevation of the top of a tower is found to be 63° 24′.7 and, from the same point, the angle of elevation of a window 56.75 ft below the top of the tower is found to be 55° 25′.4. How high is the window above the point from which the angles are measured?

54. Two telephone poles of equal height are installed on opposite sides of a road. The poles are 144 ft apart. From a point in the road the angles of elevation of the tops are found to be 41° 17′.8 and 24° 22′.6. Find the heights of the poles.

55. Two highways cross at an angle of 53° 26′. A town on one of the highways is $5\frac{3}{4}$ miles from the crossing. How far will it be between this town and another town on the other highway $7\frac{1}{4}$ miles from the crossing?

56. A radio tower is 650 ft high and is anchored by two cables in the same vertical plane. The angles made with the horizontal by the two cables are 43° 18′.3 and 57° 25′.7. Find the length of each cable and the distance between them at the foot.

57. From a point directly south of a tower the angle of elevation of its top is observed to be 48° 50′.6. From another point directly east of the tower, the angle of elevation is observed to be 42° 8′.8. What is the distance between the two points of observation if the tower is 546.5 ft high?

58. An isosceles triangle has an altitude of 18.35 and its equal sides are each 28.64. Determine the third side, the angles, and the area.

59. Two proposed highways, if extended, would meet at an angle of 44° 27′.4. The two highways are to be connected by an arc whose radius

is 4,450 ft. What is the distance from the point of intersection of the two highways to the point of tangency with the arc?

60. A 22-ft ladder rests against a wall at an angle of 62° 15′.8. How far down the wall will the top of the ladder fall if the foot is drawn away 28 in.?

61. A ship is sailing N 35° 18′.7 E at the rate of 27.5 mph. What is the ship's rate in a northerly direction; in an easterly direction?

Chapter 3

FUNCTIONS OF ANGLES OF ANY MAGNITUDE

IN CHAPTERS 1 and 2 we have dealt with the functions of angles from 0° to 90° and the solutions of the right triangle and have done nothing with angles larger than 90° except to indicate that there are such angles. Now we come to the point where we must consider angles of any magnitude and must be able to determine their functions because of the fact that such angles occur in many technical problems. In the alternating-current circuit, for instance, conditions of leading and lagging power factor require a knowledge of the functions of angles of any size and so we must extend our study to include these values. Further, there are oblique triangles that cannot be solved conveniently by merely dividing them into right triangles, and, when such is the case, a knowledge of angles larger than 90° is essential.

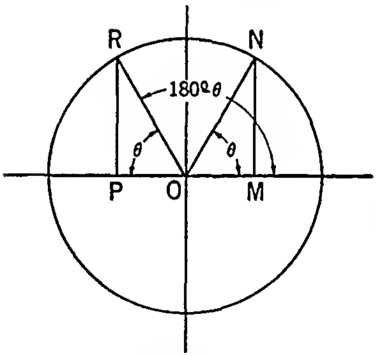


Fig. 3–1.

1. Functions of angles in second quadrant. In Fig. 3–1, the angles MON and POR are equal acute angles, each given the symbol θ. Also the triangles OMN and OPR are constructed as right triangles by making the sides NM and RP perpendicular to the horizontal axis. Therefore, from construction the triangles OPR and OMN are equal triangles, and thus

$$PR = MN,$$
$$OR = ON,$$
$$OP = OM.$$

Since angles MON and POR have each been called θ, then angle MOR is $(180° - \theta)$.

Previously we have learned that the functions of an angle in the first quadrant are ratios of the sides of the triangle. Therefore, if we take the angle θ in Fig. 3–1, we have

$$\sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{MN}{ON},$$

and

$$\cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{OM}{ON}.$$

But when we try to apply this same rule to the angle MOR or $(180° - \theta)$, then we are in difficulty immediately. The terms *side opposite* and *side adjacent* have no meaning to us. In order to overcome this difficulty we shall have to go back to our first quadrant and use a slightly different terminology. By an inspection of Fig. 3–1, it may be seen that the line MN is parallel to the y-axis and can be termed ordinate. Likewise, the line OM is parallel to the x-axis and can be termed abscissa. Again, the line ON, being the radius of the circle, can be termed distance since it is the distance of the point N from the center of the circle. Now our functions for the angle θ become:

$$\sin\theta = \frac{\text{ordinate}}{\text{distance}} = \frac{MN}{ON}, \qquad \cot\theta = \frac{\text{abscissa}}{\text{ordinate}} = \frac{OM}{MN},$$
$$\cos\theta = \frac{\text{abscissa}}{\text{distance}} = \frac{OM}{ON}, \qquad \sec\theta = \frac{\text{distance}}{\text{abscissa}} = \frac{ON}{OM},$$
$$\tan\theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{MN}{OM}, \qquad \csc\theta = \frac{\text{distance}}{\text{ordinate}} = \frac{ON}{MN}.$$

By applying these terms to the angle MOR, which equals $180° - \theta$, it becomes evident at once that they will fit any second-quadrant angle as well as the first-quadrant angle.

Therefore, we have the following:

$$\sin(180° - \theta) = \frac{\text{ordinate}}{\text{distance}} = \frac{PR}{OR}, \qquad \cot(180° - \theta) = \frac{\text{abscissa}}{\text{ordinate}} = \frac{OP}{PR},$$
$$\cos(180° - \theta) = \frac{\text{abscissa}}{\text{distance}} = \frac{OP}{OR}, \qquad \sec(180° - \theta) = \frac{\text{distance}}{\text{abscissa}} = \frac{OR}{OP},$$
$$\tan(180° - \theta) = \frac{\text{ordinate}}{\text{abscissa}} = \frac{PR}{OP}, \qquad \csc(180° - \theta) = \frac{\text{distance}}{\text{ordinate}} = \frac{OR}{PR}.$$

From our rectangular coordinates, we know that vertical lines below the horizontal axis are negative, and vertical lines above the horizontal axis are positive. Likewise, horizontal lines to the left of the vertical axis are negative and horizontal lines to the right of the vertical axis are positive. Therefore, an inspection of Figure 3–1 will show that:

$$PR = MN,$$
$$OP = -OM,$$
$$OR = ON,$$

and from these equalities we may rewrite our functions in the following form:

$$\sin(180° - \theta) = \frac{PR}{OR} = \frac{MN}{ON} = \sin\theta,$$

$$\cos(180° - \theta) = \frac{OP}{OR} = \frac{-OM}{ON} = -\frac{OM}{ON} = -\cos\theta,$$

$$\tan(180° - \theta) = \frac{PR}{OP} = \frac{MN}{-OM} = -\frac{MN}{OM} = -\tan\theta,$$

$$\cot(180° - \theta) = \frac{OP}{PR} = \frac{-OM}{MN} = -\frac{OM}{MN} = -\cot\theta,$$

$$\sec(180° - \theta) = \frac{OR}{OP} = \frac{ON}{-OM} = -\frac{ON}{OM} = -\sec\theta,$$

$$\csc(180° - \theta) = \frac{OR}{PR} = \frac{ON}{MN} = \csc\theta.$$

Therefore, it is possible to determine the functions of any angle in the second quadrant by obtaining the functions of its supplementary angle in the first quadrant and attaching the proper numerical sign.

EXAMPLE 3-1. Find the functions for an angle of 150°.

Solution:

$$\sin 150° = \sin(180° - 30°) = \sin 30°;$$

therefore, $\sin 150° = +\frac{1}{2}.$

$$\cos 150° = \cos(180° - 30°) = -\cos 30°;$$

therefore, $\cos 150° = -\frac{\sqrt{3}}{2} = -0.866.$

$$\tan 150° = \tan(180° - 30°) = -\tan 30°;$$

therefore, $\tan 150° = -\frac{\sqrt{3}}{3} = -0.577.$

$$\cot 150° = \cot(180° - 30°) = -\cot 30°;$$

therefore, $\cot 150° = -\sqrt{3} = -1.732.$

2. Functions of angles in third and fourth quadrants. In Fig. 3-2, the same procedure is followed as in Fig. 3-1 and, by using the same line of reasoning, it is evident that the triangles POR, MON, and MOQ are equal. Therefore, it follows that

$$OR = OQ = ON,$$
$$OP = -OM,$$
and $$PR = MQ = -MN.$$

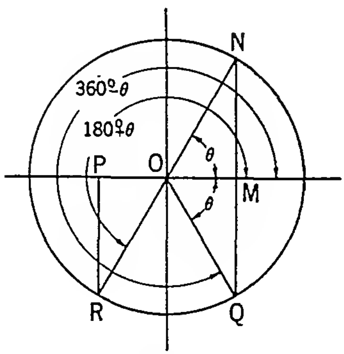


Fig. 3-2.

It is also evident that angle MOR equals $180° + \theta$, and angle MOQ equals $360° - \theta$, with the former being an angle in the third quadrant and the latter an angle in the fourth quadrant. Now since the same conditions hold true here

as in Fig. 3–1, we get the following functions for a third-quadrant angle:

$$\sin(180° + \theta) = \frac{PR}{OR} = \frac{-MN}{ON} = -\frac{MN}{ON} = -\sin\theta,$$
$$\cos(180° + \theta) = \frac{OP}{OR} = \frac{-OM}{ON} = -\frac{OM}{ON} = -\cos\theta,$$
$$\tan(180° + \theta) = \frac{PR}{OP} = \frac{-MN}{-OM} = +\frac{MN}{OM} = +\tan\theta,$$
$$\cot(180° + \theta) = \frac{OP}{PR} = \frac{-OM}{-MN} = +\frac{OM}{MN} = +\cot\theta,$$
$$\sec(180° + \theta) = \frac{OR}{OP} = \frac{ON}{-OM} = -\frac{ON}{OM} = -\sec\theta,$$
$$\csc(180° + \theta) = \frac{OR}{PR} = \frac{ON}{-MN} = -\frac{ON}{MN} = -\csc\theta.$$

Similarly, for a fourth quadrant angle, we obtain the following functions:

$$\sin(360° - \theta) = \frac{MQ}{OQ} = \frac{-MN}{ON} = -\frac{MN}{ON} = -\sin\theta,$$
$$\cos(360° - \theta) = \frac{OM}{OQ} = \frac{OM}{ON} = +\cos\theta,$$
$$\tan(360° - \theta) = \frac{MQ}{OM} = \frac{-MN}{OM} = -\frac{MN}{OM} = -\tan\theta,$$
$$\cot(360° - \theta) = \frac{OM}{MQ} = \frac{OM}{-MN} = -\frac{OM}{MN} = -\cot\theta,$$
$$\sec(360° - \theta) = \frac{OQ}{OM} = \frac{ON}{OM} = +\sec\theta,$$
$$\csc(360° - \theta) = \frac{OQ}{MQ} = \frac{ON}{-MN} = -\frac{ON}{MN} = -\csc\theta,$$

From these equations it is possible to determine the functions of any angle by simply referring that angle back to an angle in the first quadrant, the functions of which will be found in any set of tables. The following chart summarizes the signs of the functions in the four quadrants:

	Quadrant			
	I	II	III	IV
sine	+	+	−	−
cosine	+	−	−	+
tangent	+	−	+	−
cotangent	+	−	+	−
secant	+	−	−	+
cosecant	+	+	−	−

Note that all functions are positive in the first quadrant; only the sine and cosecant are positive in the second; only the tangent and cotangent are positive in the third; and only the cosine and secant are positive in the fourth.

EXAMPLE 3–2. Determine the functions for angles of 210° and 300°.

Solution: Since 210° is in the third quadrant, the functions may be found by using the functions of $(180° + \theta)$. The secant and cosecant are omitted, since they are reciprocals of the cosine and sine, respectively.

$$\sin 210° = \sin (180° + 30°) = -\sin 30°,$$
$$\sin 210° = -\frac{1}{2};$$
$$\cos 210° = \cos (180° + 30°) = -\cos 30°,$$
$$\cos 210° = -\frac{\sqrt{3}}{2} = -0.866;$$
$$\tan 210° = \tan (180° + 30°) = +\tan 30°,$$
$$\tan 210° = +\frac{\sqrt{3}}{3} = +0.577;$$
$$\cot 210° = \cot (180° + 30°) = +\cot 30°,$$
$$\cot 210° = +\sqrt{3} = +1.732.$$

Since 300° is in the fourth quadrant, the functions may be determined by using the functions of $(360° - \theta)$.

$$\sin 300° = \sin (360° - 60°) = -\sin 60°,$$
$$\sin 300° = -\frac{\sqrt{3}}{2} = -0.866;$$
$$\cos 300° = \cos (360° - 60°) = +\cos 60°,$$
$$\cos 300° = +\frac{1}{2};$$
$$\tan 300° = \tan (360° - 60°) = -\tan 60°,$$
$$\tan 300° = -\sqrt{3} = -1.732;$$
$$\cot 300° = \cot (360° - 60°) = -\cot 60°,$$
$$\cot 300° = -\frac{\sqrt{3}}{3} = -0.577.$$

At this point it will be well to take note of a very common error that causes a great deal of trouble. Many students, seeing the expressions $\sin (180° - \theta)$, $\sin (180° + \theta)$, or similar expressions, assume at once that it means that the sin is to be multiplied by $(180° - \theta)$ or $(180° + \theta)$ as the case may be, and thus get the erroneous results $\sin 180° - \sin \theta$ or $\sin 180° + \sin \theta$. *The expression* $\sin (180° - \theta)$ *does not mean the same as* $\sin 180° - \sin \theta$ *and should never be considered as the same thing.* Sin $(180° - \theta)$ means only the sine of an angle even though that angle is represented as the difference of two other angles and *the sine must not be multiplied into the* $(180 - \theta)$.

EXERCISE 3–1

Determine the sin, cos, tan, and cot of each of the angles in the following problems by referring them to first quadrant angles:

1. 176° 25′.
2. 95° 17′.
3. 200° 15′.
4. 348° 12′.
5. 117° 50′.
6. 345° 36′.
7. 260° 45′.
8. 145° 46′.
9. 280° 17′.
10. 120°.

Determine the angles that have each of the following functions:

11. sin $=+0.7824$.	**16.** tan $=+0.438$.	**21.** tan $=-\ 2.458$.
12. cot $=+1.543$.	**17.** sin $=-0.707$.	**22.** sin $=-\ 0.866$.
13. cos $=-0.3851$.	**18.** cos $=+0.866$.	**23.** sin $=\ 0.8632$.
14. tan $=-1.732$.	**19.** cos $=+0.754$.	**24.** cos $=+0.7071$.
15. tan $=-0.5670$.	**20.** cot $=-0.261$.	**25.** cot $=+\ 1.000$.

3. Functions of negative angles. We have learned that negative angles may be constructed by considering the generating line as rotating in a clockwise direction. Therefore, in Fig. 3–3, if the generating line OM rotates in a clockwise direction to the position OQ, then the angle MOQ is a negative angle $(-\theta)$. The equivalent positive angle is shown as $(+\theta)$.

Since the triangles MON and MOQ are equal, the sides also must be equal. Therefore, we may say that in Fig. 3–3

$$MQ = -MN,$$
$$OM = OM,$$
$$OQ = ON.$$

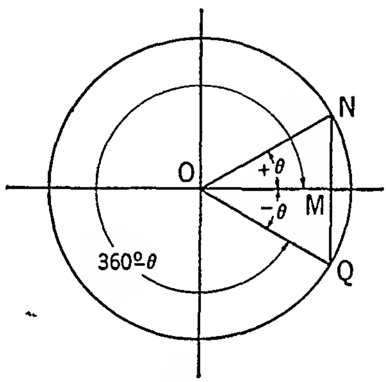


Fig. 3–3.

Then the functions of $(-\theta)$ from Fig. 3–3 are as follows:

$$\sin(-\theta) = \frac{MQ}{OQ} = \frac{-MN}{ON} = -\sin\theta, \qquad \cot(-\theta) = \frac{OM}{MQ} = \frac{OM}{-MN} = -\cot\theta,$$

$$\cos(-\theta) = \frac{OM}{OQ} = \frac{OM}{ON} = +\cos\theta, \qquad \sec(-\theta) = \frac{OQ}{OM} = \frac{ON}{OM} = +\sec\theta,$$

$$\tan(-\theta) = \frac{MQ}{OM} = \frac{-MN}{OM} = -\tan\theta, \qquad \csc(-\theta) = \frac{OQ}{MQ} = \frac{ON}{-MN} = -\csc\theta.$$

It can be shown also that these same conditions hold true for negative angles in any quadrant.

Example 3–3. Determine the functions of $-30°$ and $-120°$.

Solution:

$$\sin -30° = -\sin 30° = -\frac{1}{2}$$

$$\cos -30° = +\cos 30° = +\frac{\sqrt{3}}{2}$$

$$\tan -30° = -\tan 30° = -\frac{\sqrt{3}}{3}$$

$$\cot -30° = -\cot 30° = -\sqrt{3}$$

$$\sin -120° = -\sin 120° = -\sin(180° - 60°) = -(+\sin 60°) = -\frac{\sqrt{3}}{2}$$

$$\cos -120^\circ = +\cos 120^\circ = +\cos (180^\circ - 60^\circ) = +(-\cos 60^\circ) = -\frac{1}{2}$$

$$\tan -120^\circ = -\tan 120^\circ = -\tan (180^\circ - 60^\circ) = -(-\tan 60^\circ) = +\sqrt{3}$$

$$\cot -120^\circ = -\cot 120^\circ = -\cot (180^\circ - 60^\circ) = -(-\cot 60^\circ) = +\frac{\sqrt{3}}{3}$$

EXERCISE 3–2

Express the trigonometric functions of the following negative angles as functions of positive angles in the first quadrant:

1. -154°.
2. $-221^\circ\ 15'$.
3. $-216^\circ\ 30'$.
4. $-342^\circ\ 18'$.
5. $-\frac{\pi}{4}$.
6. $-105^\circ\ 47'$.
7. $-196^\circ\ 18'\ 15''$.
8. $-\frac{\pi}{8}$.
9. $-327^\circ\ 17'$.
10. $-286^\circ\ 36'$.
11. -1.438.
12. $-174^\circ\ 16'$.

Determine the values of the trigonometric functions of the following angles:

13. $-82^\circ\ 32'$.
14. $-\frac{3\pi}{2}$.
15. $-12^\circ\ 30'$.
16. $-97^\circ\ 43'$.
17. $-168^\circ\ 41'$.
18. $-185^\circ\ 12'$.
19. $-212^\circ\ 10'$.
20. $-\frac{4\pi}{3}$.
21. $-342^\circ\ 5'$.
22. $-265^\circ\ 18'$.
23. $-27^\circ\ 37'$.
24. $-308^\circ\ 47'$.

4. Functions of angles differing by 90°. In the preceding articles we have considered angles differing by 180° or by 360° and we have found that a function of the one angle may be expressed in terms of the same function of the other angle. Now let us consider angles that differ in value by 90°.

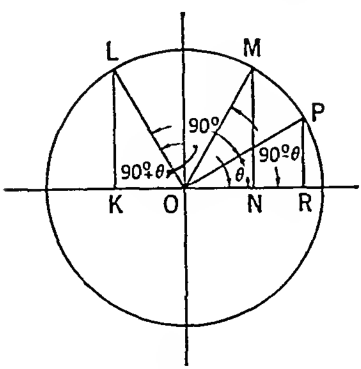


Fig. 3–4.

In Fig. 3–4, the angles POR, MON, and LOK have been constructed so that, if POR is θ, then MON is $(90^\circ - \theta)$ and LOR is $(90^\circ + \theta)$. This construction makes the sides of triangle LOK perpendicular to the sides of triangle POR and, from geometry, the two triangles are equal. Therefore,

$$KL = OR, \quad OK = -RP, \quad OL = OP.$$

From this we note that:

$$\sin (90 + \theta) = \frac{KL}{OL} = \frac{OR}{OP} = \cos \theta,$$

$$\cos (90 + \theta) = \frac{OK}{OL} = \frac{-RP}{OP} = -\sin \theta,$$

$$\tan (90 + \theta) = \frac{KL}{OK} = \frac{OR}{-RP} = -\cot \theta,$$

$$\cot (90 + \theta) = \frac{OK}{KL} = \frac{-RP}{OR} = -\tan \theta,$$
$$\sec (90° + \theta) = \frac{OL}{OK} = \frac{OP}{-RP} = -\csc \theta,$$
$$\csc (90° + \theta) = \frac{OL}{KL} = \frac{OP}{OR} = +\sec \theta.$$

Also, since the angle POR equals θ and the angle MON equals $(90° - \theta)$, then these two angles are complementary angles and their cofunctions are equal. Therefore,

$$\sin (90 - \theta) = \cos \theta,$$
$$\cos (90 - \theta) = \sin \theta,$$
$$\tan (90 - \theta) = \cot \theta,$$
$$\cot (90 - \theta) = \tan \theta,$$
$$\sec (90 - \theta) = \csc \theta,$$
$$\csc (90 - \theta) = \sec \theta.$$

It can be shown that these eight relationships hold true for angles in any quadrant and therefore we may say that angles that differ by 90° have their cofunctions equal in absolute value.

EXERCISE 3-3

Determine the functions of the following angles in quadrant 2, by use of the relationships of Article 4:

1. 150°.	3. 135°.	5. 165°.	7. 179° 12′.	9. 140° 25′.
2. 120°.	4. 105°.	6. 97° 16′.	8. 115° 40.	10. 103° 50′.

5. Functions as lines. Just as functions of angles in the first quadrant were represented as lines in a unit circle, so may the functions of angles in all quadrants be represented. In Fig. 3-5, an angle in the second quad-

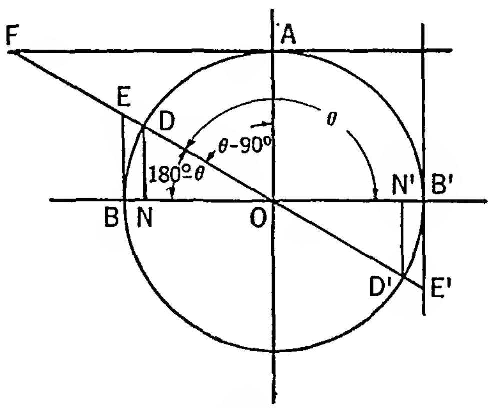


Fig. 3-5.

rant is shown in a circle with a unit radius. Therefore, by construction,

$$OD = OB = OA = 1,$$
$$OD' = OB' = 1.$$

Triangles DON and EOB are similar.
Triangles $D'ON'$ and $E'OB'$ are similar.
Triangle DON = triangle $D'ON'$.
Triangle EOB = triangle $E'OB'$.

Now the functions of angle θ will be as follows:

$$\sin\theta = \frac{ND}{OD} = \frac{ND}{1} = ND,$$
$$\cos\theta = \frac{ON}{OD} = \frac{ON}{1} = ON,$$
$$\tan\theta = \frac{ND}{ON} = \frac{N'D'}{ON'} = \frac{B'E'}{OB'} = \frac{B'E'}{1} = B'E',$$
$$\cot\theta = \tan(\theta - 90°) = \frac{AF}{OA} = \frac{AF}{1} = AF,$$
$$\sec\theta = \frac{OD}{ON} = \frac{OD'}{ON'} = \frac{OE'}{OB'} = \frac{OE'}{1} = OE',$$
$$\csc\theta = \sec(\theta - 90°) = \frac{OF}{OA} = \frac{OF}{1} = OF.$$

It will be noticed that a tangent line at the right of the circle and another at the top of the circle have been used to determine the tangent and cotangent respectively. This is necessary in order that the direction of the line shall indicate the sign of the function involved.

For example, $\tan\theta = BE$ as well as $B'E'$, but in a rectangular coördinate system the line BE is a positive line and the line $B'E'$ is a negative line, and, since the tangent is negative in the second quadrant, its true representation in a unit circle is the line $B'E'$, which is found by extending the terminal side back to meet the tangent line at the right of the circle.

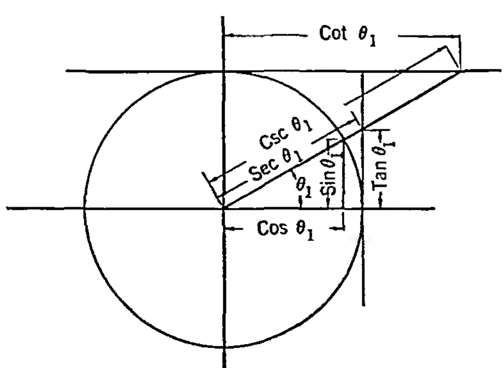


Fig. 3-6.

Again, in order to obtain an expression in each case with the unit radius as a denominator, it has been necessary to resort to the cofunction of the complementary angle for cot θ and csc θ.

The numerical signs for the secant and cosecant are not easily determined from the direction of their representative lines since these lines lie along the radius of the circle which is always considered positive when measured outward from the origin. However, since they are reciprocals of the cosine and sine, respectively, their signs will be the same.

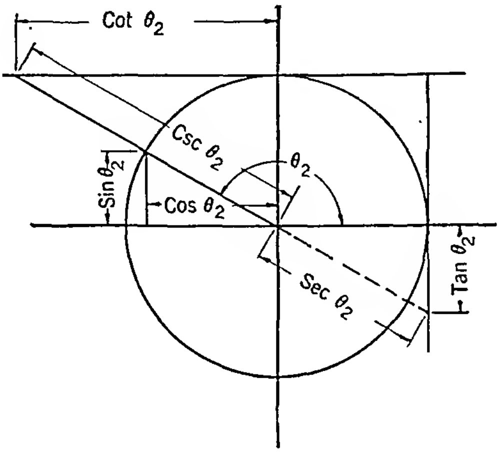


Fig. 3–7.

Now, since each of the six functions is actually equal to the length of some definite line in a unit circle, Fig. 3–5 can be redrawn as in Fig. 3–7 to show the functions of second-quadrant angles as lines. Fig. 3–6 also has been drawn to show the various functions as lines for angles in the first quadrant.

In a similar way, the functions of angles in the third and fourth quadrants may be represented as lines in a unit circle. Fig. 3–8 shows the func-

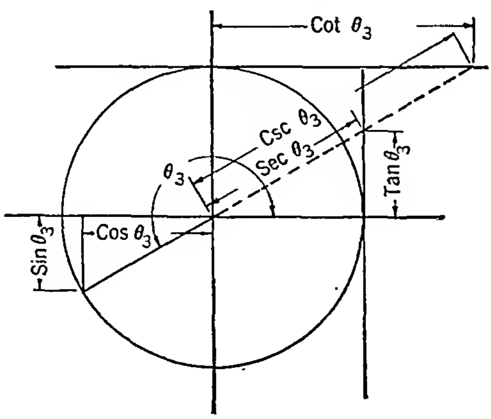


Fig. 3–8.

tions for an angle in quadrant 3, and Fig. 3–9 shows the functions for an angle in quadrant 4.

For all quadrants the tangent is determined by extending the terminal side of the angle to meet the tangent line drawn at the right side of the circle so that the correct sign is always obtained. Likewise the cotangent is determined from the line drawn tangent to the circle at the top. Tangent

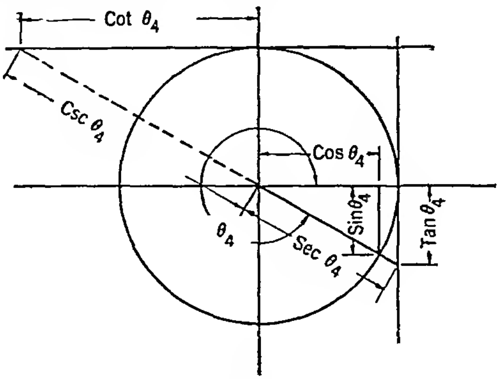


Fig. 3–9.

lines at the left or bottom of the circle are not used because they do not always give the correct signs.

6. Variation in the functions. In Chapter 1 we have shown how the six functions vary as the angle changes from zero to 90°. A study of Figs. 3–6 to 3–9 inclusive will show the variation of the functions as the angle continues on from 0° to 360°. The following chart summarizes these conditions:

As angle θ varies from	$\sin\theta$ varies from	$\cos\theta$ varies from	$\tan\theta$ varies from
0° to 90°	0 to +1	+1 to 0	0 to $+\infty$
90° to 180°	+1 to 0	0 to −1	$-\infty$ to 0
180° to 270°	0 to −1	−1 to 0	0 to $+\infty$
270° to 360°	−1 to 0	0 to +1	$-\infty$ to 0

As angle θ varies from	$\cot\theta$ varies from	$\sec\theta$ varies from	$\csc\theta$ varies from
0° to 90°	$+\infty$ to 0	+1 to $+\infty$	$+\infty$ to +1
90° to 180°	0 to $-\infty$	$-\infty$ to −1	+1 to $+\infty$
180° to 270°	$+\infty$ to 0	−1 to $-\infty$	$-\infty$ to −1
270° to 360°	0 to $-\infty$	$+\infty$ to +1	−1 to $-\infty$

EXERCISE 3–4

1. Construct a unit circle with a radius of 1 in. and indicate on this circle the six functions of an angle of 135°.
2. Repeat Problem 1 for an angle of 225°.
3. Repeat Problem 1 for an angle of 315°.

7. Given one function of an angle to find the other functions. If one function of an angle is given, it is possible to construct the angle or angles that satisfy the value of this function and to determine the other functions.

EXAMPLE 3–4. Determine the other functions of the angle whose sine $= -\frac{3}{5}$.

Solution: Since $\sin\theta$ is negative there will be *two angles, one in the third* quadrant and one in the fourth quadrant, that fit the given condition. Construction will show the triangles to be like Fig. 3–10.

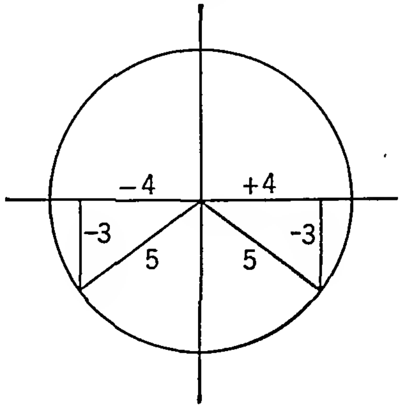


Fig. 3–10.

The other leg of the triangle $= \pm\sqrt{5^2 - 3^2} = \pm\sqrt{16} = \pm 4$.
Now the other functions will be

$$\cos\theta = -\frac{4}{5} \text{ or } +\frac{4}{5}, \quad \cot\theta = +\frac{4}{3} \text{ or } -\frac{4}{3},$$
$$\tan\theta = +\frac{3}{4} \text{ or } -\frac{3}{4}, \quad \sec\theta = -\frac{5}{4} \text{ or } +\frac{5}{4},$$
$$\csc\theta = -\frac{5}{3}.$$

EXERCISE 3–5

Given the following functions of a central angle, determine the other functions and the quadrants in which the angles lie:

1. $\sin A = -\frac{2}{3}$.
2. $\cos A = +\frac{3}{4}$.
3. $\tan A = -\frac{4}{5}$.
4. $\cot A = +\frac{3}{8}$.
5. $\sec A = -2$.
6. $\sin B = +\frac{4}{5}$.
7. $\cos B = -\frac{3}{5}$.
8. $\tan B = +\frac{2}{3}$.
9. $\cot B = -\frac{3}{8}$.
10. $\csc B = +2$.

11. If $\tan A = \sqrt{2}$, how large is $\cot A$? What are the other functions of A?

12. What angle will have a sine equal to the product $2 \sin 30° \cos 30°$? The product $2 \sin 45° \cos 45°$?

8. Graphs of the functions. A very easy way in which to visualize the variations that the functions go through is to make a graph for each one. If we let y equal the particular function desired, then we have an equation that can be plotted in rectangular coördinates.

EXAMPLE 3–5. Plot the graph of the equation $y = \sin \theta$.

Solution: Using values along the abscissa for the angles from 0° to 360° and values along the ordinate for $\sin \theta$ gives the curve of Fig. 3–11.

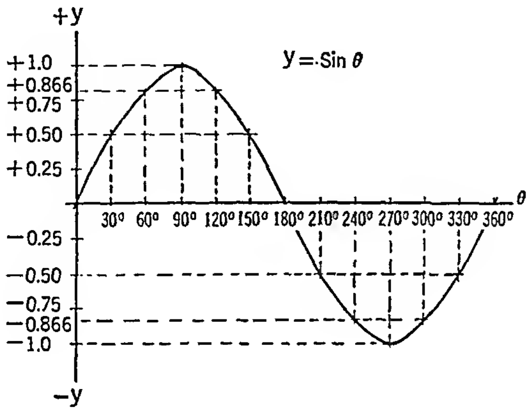


Fig. 3–11.

A continuation of this curve either in the positive or negative directions (right or left) will show a repetition of the figure.

A study of this figure will show that the sine is positive in the first two quadrants (0° to 180°) and negative in the third and fourth quadrants (180° to 360°). Also, the sine varies in absolute value between zero and unity.

EXAMPLE 3–6. Plot the graph of $y = \tan \theta$.

Solution: Plotting $\tan \theta$ along the y-axis and the values for the angle θ along the x-axis gives the curve of Fig. 3–12.

9. Periodicity of the functions. A study of the variations of the sine in Article 6 and in the graphs of Article 8 will show that, as the angle varies from 0° to 360°, the sine varies through all its possible values and returns at 360° to the same value with which it started at 0°. If the angle is continued from 360° to 720°, the sine will show a repetition of the values that it had from 0° to 360° and for each succeeding 360° the sine will repeat these values. For this reason the sine is called a periodic function of 2π radians, which is 360°, or, the period of the sine is 2π. It is evident then that *the period of a function is determined by the point at which the values begin to repeat.*

A study of the graph of the tangent in Example 3–5 will show that the period of the tangent is 180° or π since the values of the tangent begin to repeat after the angle has passed through 180°.

The period of a function of an angle, then, generally can be found by plotting the graph of the function. Some angles do not lend themselves readily to plotting, and a method for finding the periods of the functions of such angles is shown in Example 3–7.

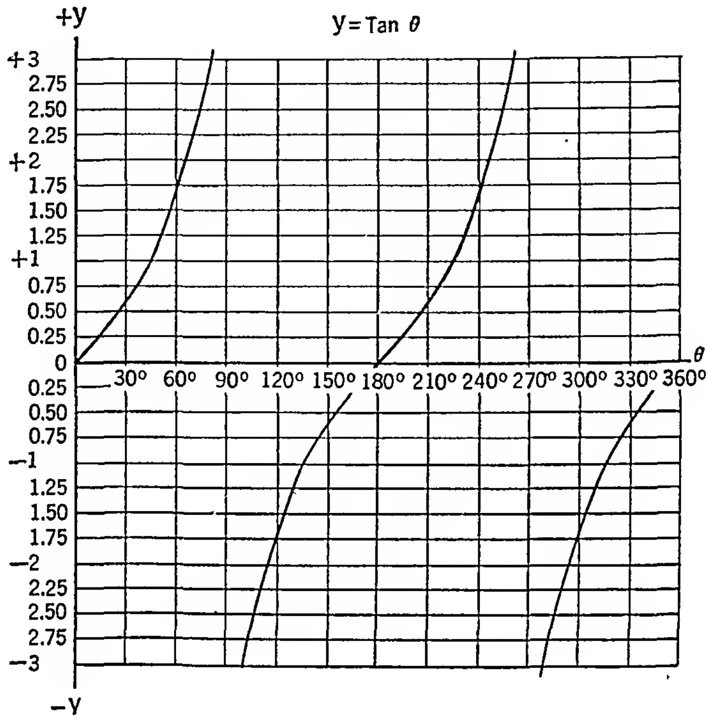


Fig. 3–12.

EXAMPLE 3–7. Find the period of $\sin \frac{2x}{3}$.

Solution: Since the values of $\sin \theta$ begin to repeat after θ has varied through 360°, or from 0° to 2π, then the values of $\sin \frac{2x}{3}$ must also begin to repeat when $\frac{2x}{3}$ has varied from 0° to 2π. Therefore setting $\frac{2x}{3}$ equal to 0° and also equal to 2π, and solving these two resulting equations will give the values for x that will determine the period of $\sin \frac{2x}{3}$. Thus,

$$\frac{2x}{3} = 0^\circ$$

or

$$x = 0^\circ;$$

and

$$\frac{2x}{3} = 2\pi$$

or

$$x = \frac{6\pi}{2} = 3\pi.$$

Therefore, x varies from 0° to 3π and the period of $\sin \frac{2x}{3} = 3\pi$. The graph of the function is shown in Fig. 3-13.

A chart first should be made up in order to plot the graph of $\sin \frac{2x}{3}$.

x	$\frac{2x}{3}$	$y = \sin \frac{2x}{3}$
0°	0°	0
30°	20°	0.342
60°	40°	0.643
90°	60°	0.866
120°	80°	0.985
150°	100°	0.985
180°	120°	0.866
210°	140°	0.643
240°	160°	0.342
270°	180°	0
300°	200°	−0.342
330°	220°	−0.643
360°	240°	−0.866
390°	260°	−0.985
420°	280°	−0.985
450°	300°	−0.866
480°	320°	−0.643
510°	340°	−0.342
540°	360°	0

In plotting the graph, the values of x in column 1 of the chart should be plotted against the values of y in column 3.

It should be noted that the wave $y = \sin \frac{2x}{3}$ does not begin to repeat values until the value of x has reached 540°, or 3π, which is the period of $\sin \frac{2x}{3}$.

EXERCISE 3-6

1. Plot the graph of $y = \cos \theta$ and determine the period of the cos.
2. Plot the graph of $y = \cot \theta$ and determine the period of the cot.

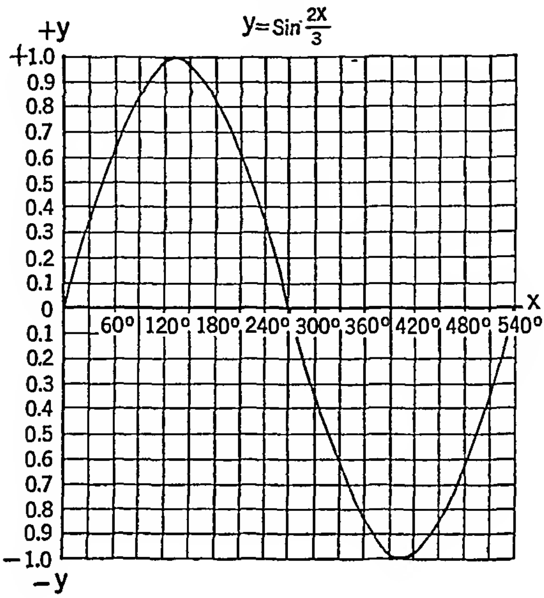


Fig. 3–13.

3. Plot the graph of $y = \sec\theta$ and determine the period of the see.
4. Plot the graph of $y = \csc\theta$ and determine the period of the csc.
5. Plot the graph of $y = \cos 2\theta$ and determine the period of $\cos 2\theta$.
6. Plot the graph of $y = \sin\frac{\theta}{3}$ and determine the period of $\sin\frac{\theta}{3}$.
7. Plot the graph of $y = \tan\frac{\pi x}{4}$ and determine the period of $\tan\frac{\pi x}{4}$.
8. Plot the graph of $y = \sec\frac{3x}{2}$ and determine the period of $\sec\frac{3x}{2}$.

10. Inverse functions. Often it is convenient to write an angle in terms of one of its functions instead of writing the angle in its actual value in degrees or in radians.

Thus in the equation $y = \sin\theta$, θ is an angle whose sine is y. When the angle is stated in terms of the functions in this way, *the expression is called an inverse function and is written in mathematical symbols as* $\theta = \text{arc}\sin y$ *or* $\theta = \sin^{-1} y$. This latter symbol may look like an exponent but it should be understood that the -1 is a part of the symbol $\sin^{-1} y$ and not an exponent and it should be read as "the inverse sine y," or better still as "*the angle whose sine is y*," which emphasizes the fact that *$\sin^{-1} y$ represents an angle.* Similarly, other inverse functions may be written as

$$\begin{array}{lcl} \cos^{-1} y & \text{or} & \text{arc}\cos y; \\ \tan^{-1} y & \text{or} & \text{arc}\tan y; \\ \cot^{-1} y & \text{or} & \text{arc}\cot y; \\ \sec^{-1} y & \text{or} & \text{arc}\sec y; \\ \csc^{-1} y & \text{or} & \text{arc}\csc y. \end{array}$$

These expressions all represent angles.

The trigonometric functions have single values; that is, if θ is given, sine θ has one value only. On the other hand, the inverse functions are multiple valued. For example, if $\theta = \sin^{-1}\frac{1}{2}$, then there are two angles that fit this expression, since $\sin 30° = \frac{1}{2}$ and $\sin 150° = \frac{1}{2}$. Therefore,

$$\sin^{-1}\frac{1}{2} = 30° \text{ or } 150°.$$

A convenient method for limiting the angle to one quadrant is to use the number of the quadrant as a small subscript. Thus, $\sin_3^{-1}\left(-\frac{1}{4}\right)$ means an angle in quadrant 3.

EXAMPLE 3–8. Find the angles represented by $\theta = \cos^{-1}\frac{1}{2}$.

Solution: Since $\cos\theta = +\frac{1}{2}$, the angle must lie in the first quadrant or in the fourth quadrant. The angle in the first quadrant whose cosine is $+\frac{1}{2}$ is 60° and the fourth quadrant angle then is $360° - 60° = 300°$.

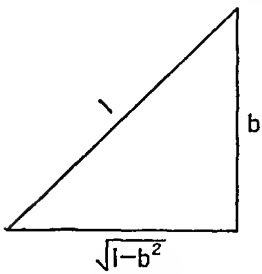


Fig. 3–14.

Therefore, $\cos^{-1}\frac{1}{2} = 60° \text{ or } 300°.$

EXAMPLE 3–9. Simplify the following:

$$\cos(\sin^{-1} b).$$

Solution: Since $\sin^{-1} b$ represents an angle, then the expression is read as "the cosine of the angle whose sin is b." If the sin is b, the side opposite the angle is b and the hypotenuse is 1. The side adjacent to the angle is $\sqrt{1-b^2}$, as shown in Fig. 3–14.

Let $\sin^{-1} b = \theta.$

Then, $\sin\theta = b$

and $\cos\theta = \pm\dfrac{\sqrt{1-b^2}}{1} = \pm\sqrt{1-b^2}.$

11. Graphs of the inverse functions. Since the equations $y = \sin\theta$ and $\theta = \sin^{-1} y$ express the same relationship between θ and y, the graph for the equation $\theta = \sin^{-1} y$ is exactly the same as the graph for the equation $y = \sin\theta$ with the exception that the axes are reversed with respect to the curve. Thus the graph of $\theta = \sin^{-1} y$ is shown in Fig. 3–15.

EXERCISE 3–7

Find all the angles from 0° to 360° that are represented by the following:

1. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.
2. $\cos^{-1}(+0.866)$.
3. $\tan^{-1}(-0.707)$.
4. $\cot^{-1}(+1.732)$.
5. $\sin^{-1}\left(+\frac{1}{\sqrt{2}}\right)$.
6. $\cos^{-1}(-0.259)$.
7. $\tan^{-1}(+1.8546)$.
8. $\cot^{-1}(-0.4627)$.
9. $\cos^{-1}(-0.7261)$.
10. $\sin^{-1}(-0.0543)$.

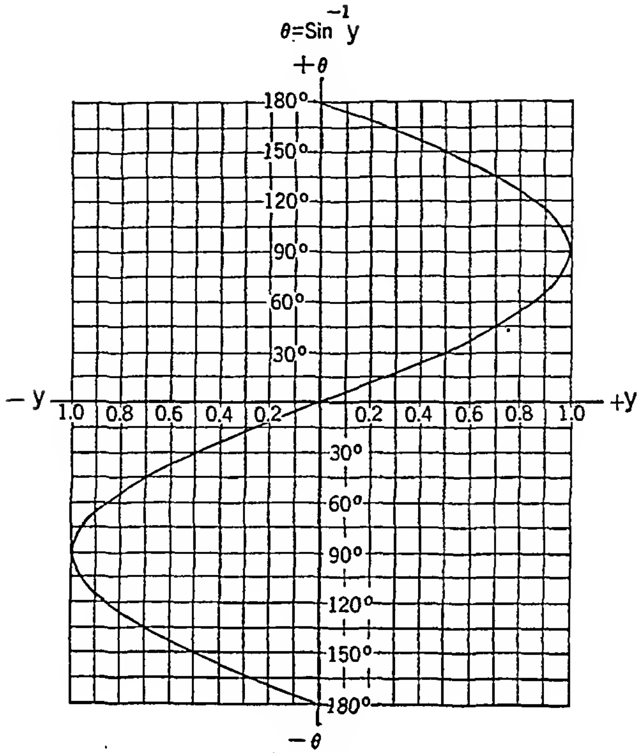


Fig. 3–15.

Simplify the following:

11. $\sin(\sin^{-1} a)$.

12. $\sin\left(\cos^{-1} - \frac{\sqrt{2}}{2}\right)$.

13. $\tan(\cot^{-1} + \sqrt{3})$.

14. $\sin^{-1}(\tan 30°)$.

15. Plot the graph of the equation $\theta = \cos^{-1} a$.

16. Plot the graph of the equation $\theta = \tan^{-1} b$.

12. Definitions of simple trigonometric equations and identities. In algebra there are two kinds of equations: *identical* and *conditional*. *An identical equation is one that holds true for all values of the quantities involved.* Thus the algebraic statement

$$a^2 + 2ab + b^2 = (a + b)^2$$

is an identical equation, or simply an identity, because it is true for any and all values of a and b.

A conditional equation is one that holds true for only certain particular values or sets of values of the quantities involved. Thus, the algebraic statement

$$x^2 - 10x + 16 = 0$$

is a conditional equation or simply an equation, since it is true only for certain values of x and is not true for any other values. In this equation x may equal 8 or 2 but no other values.

The same classifications apply to equations in which trigonometric quantities are involved. Thus the statement

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

is a trigonometric identity because it holds true for all values of the angle θ. However, the trigonometric statement

$$\sin \theta - \cos \theta = 0$$

is a trigonometric equation since it holds true only when the angle θ is equal to 45° or 225° or angles coterminal with either of these.*

It is evident then that the relationships we have found to exist between the various functions are identities and it also should be evident that an identity is merely a statement to the effect that the quantity on the left-hand side of the equation is equal to the quantity on the right-hand side of the equation for all values of the letters involved and that either quantity may replace the other or be changed into the other.

On the other hand, an equation implies the question, "For what value or values of the angle does this statement hold true?" The solution of an equation consists of the processes of solving for all the values of the unknown that satisfy the equation, which is in no way different from the solution of an algebraic equation.

13. Solution of trigonometric identities. Frequently, it becomes necessary to change a trigonometric expression from one form into another identical with it so that it can be used more conveniently. Such changes are sometimes difficult to make and no set method of attack can be prescribed. However, the student will find that it is generally best to *change the more complicated side into the simpler one and leave the simpler side unchanged.*

The fundamental identities of Chapter 1 show various changes that can be made; others may easily be developed from them. Changing the one side into terms of the sine and cosine is a useful and helpful expedient. The following cases illustrate the process:

EXAMPLE 3–10. Prove the identity $\sec \theta = \frac{\tan \theta}{\sin \theta}$.

Solution: The right-hand member being the more complicated we will work with it and leave the left-hand member as it is. Thus,

$$\frac{\tan \theta}{\sin \theta} = \frac{\frac{\sin \theta}{\cos \theta}}{\sin \theta} = \frac{\sin \theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta} = \sec \theta.$$

---

* An angle coterminal with 45° is 405°, since its terminal side coincides with the terminal side of 45°.

EXAMPLE 3–11. Prove the identity $\dfrac{1 + \sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1 - \sin\theta}$.

Solution: Since both members are equally complicated we will work with the left-hand member and transform it into the right-hand member. Let us first multiply both numerator and denominator of the left-hand member by $(1 - \sin\theta)$ in order that we may get this expression in the denominator as required by the right-hand member. Thus,

$$\left(\frac{1 + \sin\theta}{\cos\theta}\right)\left(\frac{1 - \sin\theta}{1 - \sin\theta}\right) = \left(\frac{1 - \sin^2\theta}{\cos\theta\,(1 - \sin\theta)}\right) = \frac{\cos^2\theta}{\cos\theta\,(1 - \sin\theta)} = \frac{\cos\theta}{1 - \sin\theta}.$$

EXERCISE 3–8

Prove the following identities, changing the form of one member and leaving the other unchanged:

1. $\tan^2 x \sin^2 x = \tan^2 x - \sin^2 x$.
2. $\sin x = \dfrac{\tan x}{\sqrt{1 + \tan^2 x}}$.
3. $\tan x + \cot x = \sec x \csc x$.
4. $\cos^2 x + 2\sin^2 x = 1 + \sin^2 x$.
5. $\dfrac{1 + \tan^2 x}{\csc^2 x} = \tan^2 x$.
6. $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$.
7. $\csc^2 x - \cot^2 x = 1$.
8. $\dfrac{\sqrt{1 + \tan^2\theta}}{\sqrt{1 + \cot^2\theta}} = \tan\theta$.
9. $\cos^2 x \csc^2 x = \csc^2 x - 1$.
10. $(1 + \cos\theta)(\csc\theta - \cot\theta) = \sin\theta$.
11. $\sin^2 x \cot^2 x + \cos^2 x \tan^2 x = 1$.
12. $\sec^2 x = \sin^2 x\,(\csc^2 x + \sec^2 x)$.
13. $\dfrac{1 + \tan^2\theta}{1 + \cot^2\theta} = \dfrac{\sin^2\theta}{\cos^2\theta}$.
14. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2\csc x$.
15. $2\sin x + \tan x = \dfrac{2 + \sec x}{\csc x}$.
16. $(1 - \sin x)\cos^2 x = (1 + \sin x)(1 - \sin x)^2$.
17. $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$.
18. $\dfrac{1 + \cos x}{1 - \cos x} = 1 + \dfrac{2\cos x(1 + \cos x)}{\sin^2 x}$.

14. Solution of trigonometric equations. The solution for a trigonometric equation consists of the process of finding all the angles that satisfy the equation. Usually it is enough to give the values that lie between 0° and 360°.

The method of procedure is to *change the equation until it is in terms of one function and then solve this resulting equation for this function just the same as a similar algebraic equation would be solved.* After the function has been determined, the angles can be found that fit the function. Each angle then should be substituted in the original equation to see if it satisfies that expression. Those that satisfy are retained and all others rejected. The following cases illustrate some of the methods that may be used:

EXAMPLE 3–12. Solve the equation $\cos x = 1 - \sin^2 x$.

Solution: Changing everything in the equation to the cosine function, we obtain

$$\cos x = \cos^2 x.$$

Transposing, $\cos^2 x - \cos x = 0.$

Factoring, $\cos x(\cos x - 1) = 0.$

Now since the product of the two factors is zero, either may be set equal to zero. Therefore,

$$\cos x = 0, \qquad \cos x - 1 = 0.$$

Then $x = 90°$ or $270°$ $\qquad \cos x = 1$

$$x = 0° \text{ or } 360°.$$

Substituting each of these values in the original equation, we find that each one satisfies the original. Therefore, the solutions are

$$x = 0°; 90°; 270°; 360°.$$

EXAMPLE 3–13. Solve the equation

$$\sin^2 x - \cos x = \frac{1}{4}.$$

Solution: Changing to terms of cos gives

$$1 - \cos^2 x - \cos x = \frac{1}{4}.$$

Clearing fractions, $4 - 4\cos^2 x - 4\cos x = 1.$

Therefore, $4\cos^2 x + 4\cos x - 3 = 0.$

Since this equation is a quadratic equation in $\cos x$, it may be solved by the application of the quadratic formula. Using the quadratic formula,

$$\cos x = \frac{-4 \pm \sqrt{(4)^2 - (4)(4)(-3)}}{(2)(4)},$$

$$\cos x = \frac{-4 \pm \sqrt{16 + 48}}{8} = \frac{-4 \pm 8}{8} = +\frac{1}{2} \text{ or } -\frac{3}{2},$$

$$x = 60° \text{ or } 300°.$$

By substituting these two values back in the original equation, we find that both values satisfy the expression. Therefore, for the equation

$$\sin^2 x - \cos x = \frac{1}{4},$$
$$x = 60° \text{ or } 300°.$$

EXAMPLE 3–14. Solve the following equation:

$$\sec^2 x = 1 - \sqrt{3}\tan x.$$

Solution: Changing everything to sines and cosines,

$$\frac{1}{\cos^2 x} = 1 - \sqrt{3}\,\frac{\sin x}{\cos x}.$$

Changing to common denominator,

$$1 = \frac{\cos^2 x - \sqrt{3}\sin x \cos x}{\cos^2 x}.$$

Then $\qquad \cos^2 x - 1 = \sqrt{3}\sin x \cos x$

or $\qquad \cos^2 x - 1 = \sqrt{3}\sqrt{1 - \cos^2 x}\cos x.$

Squaring, $\cos^4 x - 2\cos^2 x + 1 = 3\cos^2 x(1 - \cos^2 x) = 3\cos^2 x - 3\cos^4 x.$

Collecting terms, $\qquad 4\cos^4 x - 5\cos^2 x + 1 = 0.$

Factoring, $\qquad (4\cos^2 x - 1)(\cos^2 x - 1) = 0.$

Then

$$4\cos^2 x - 1 = 0, \qquad \cos^2 x - 1 = 0,$$
$$4\cos^2 x = 1, \qquad \cos^2 x = 1,$$
$$\cos^2 x = \frac{1}{4}, \qquad \cos x = \pm 1,$$
$$x = 0°;\ 180°;\ 360°.$$
$$\cos x = \pm\frac{1}{2},$$
$$x = 60°;\ 120°;\ 240°;\ 300°.$$

By substituting these values in the original equation we find

For 0°: $\qquad (1)^2 = 1 - (\sqrt{3})(0),$
$\qquad 1 = 1.$

For 60°: $\qquad (2)^2 = 1 - (\sqrt{3})(\sqrt{3}),$
$\qquad 4 = 1 - 3.$ This does not apply.

For 120°: $\qquad (-2)^2 = 1 - (\sqrt{3})(-\sqrt{3}),$
$\qquad 4 = 1 + 3 = 4.$

For 180°: $\qquad (-1)^2 = 1 - (\sqrt{3})(0),$
$\qquad 1 = 1.$

For 240°: $\qquad (-2)^2 = 1 - (\sqrt{3})(\sqrt{3}),$
$\qquad 4 = 1 - 3.$ This does not apply.

For 300°: $\qquad (-2)^2 = 1 - (\sqrt{3})(-\sqrt{3}),$
$\qquad 4 = 1 + 3 = 4.$

For 360°: $\qquad (1)^2 = 1 - (\sqrt{3})(0),$
$\qquad 1 = 1.$

Then the values that fit are 0°; 120°; 180°; 300°; 360°.

From this it is evident that the process of squaring introduces additional angles that may not fit the original condition, and, therefore, all angles found in this way must be carefully checked.

EXERCISE 3-9

Solve the following trigonometric equations:

1. $\sin^2 x = \frac{1}{4}$.
2. $\cos^2 x = \frac{1}{4}$.
3. $2 \cos x = \sec x$.
4. $2 \sin^2 \theta + \sin \theta - 3 = 0$.
5. $2 \tan^2 \theta = \sec^2 \theta$.
6. $(\cot \theta + 1)(3 \sec^2 \theta - 4) = 0$.
7. $3 \tan^2 x + 5 \sec x + 4 = 0$.
8. $3 \cos^2 \theta - 2 \sin^2 \theta + 6 \cos \theta = 0$.
9. $1 - \sin x = \sqrt{3} \cos x$.
10. $2 \sin x + \cot x = 1 + 2 \cos x$.
11. $\cos^2 x - \sin^2 x = \sin x$.
12. $2 \cot x + \cos x \cot x = 2 \cot x \sin^2 x$.

REVIEW EXERCISE 3-10

Prove the following identities:

1. $\frac{1}{\csc^2 x} + \frac{1}{\sec^2 x} = \cot x \tan x$.
2. $\sin^2 x = \frac{1}{\cot^2 x + 1}$.
3. $\sin x \sec x \cot x = 1$.
4. $\sec^2 x + \cot^2 x = \tan^2 x + \csc^2 x$.
5. $\sin x \cos x + \tan x \sin^2 x = \tan x$.
6. $\tan x = \cot x + \frac{1 - 2 \cos^2 x}{\sin x \cos x}$.
7. $\sec^2 x = \sin^2 x \sec^2 x + 1$.
8. $\tan^2 x \cos^2 x = 1 - \sin^2 x \cot^2 x$.
9. $(\tan x + \cot x)^2 = \sec^2 x + \csc^2 x$.
10. $\sin x + \cos x = (\sin x \cos x)(\sec x + \csc x)$.
11. $\sec^4 x - \tan^4 x = 2 \tan^2 x + 1$.
12. $\cos x (\cot x + \tan x) = \csc x$.
13. $\sin x (1 + \tan x) - \csc x = \sec x - \cos x(1 + \cot x)$.
14. $\frac{1 + \cos x}{1 - \cos x} = \frac{\sin^2 x}{(1 - \cos x)^2}$.
15. $\frac{\cos x}{1 - \tan x} - \cos x = \sin x - \frac{\sin x}{1 - \cot x}$.

Solve the following equations:

16. $\sin x = \tan x$.
17. $4 \sin^2 x + 2 \cos^2 x = 3$.
18. $2 \cos^2 \theta - (1 - \sin^2 \theta) = \frac{3}{4}$.
19. $\sin^2 x - \cos x = \frac{1}{4}$.
20. $7 \tan^2 x - 3 = \sec^2 x$.
21. $\tan x = -\frac{\sqrt{2}}{\sec x}$.
22. $\tan x = \frac{2}{3} \cos x$.
23. $\sin x + \cos x \cot x - 2 = 0$.
24. $2 \cos x + 2 \sec x = 5$.
25. $\tan x \sin x = \frac{3}{2}$.
26. $1 - 3 \sin x = 2(\sin^2 x + 1)$.
27. $\tan^2 x - 4 + 3 \cot^2 x = 0$.
28. $4 \sin^2 x + 1 = 6 \tan^2 x$.
29. $5 \sin x + 2 \cos^2 x - 4 = 0$.
30. $\tan x = \frac{\cot x - 1}{\cot x + 1}$.

Chapter 4

FUNCTIONS OF TWO ANGLES

IT IS important at times to be able to determine the functions of the sum or difference of two angles in terms of the functions of the separate angles. At first glance, it would seem that the sine of the sum of two angles should be equal to the sum of the sines of each of the angles, but a closer inspection will show that this, in general, is not true. To illustrate, let $x = 30°$ and $y = 60°$. Then

$$\sin x + \sin y = \sin 30° + \sin 60° = 0.5 + 0.866 = 1.366;$$

but $$\sin (x + y) = \sin (30 + 60) = \sin 90° = 1.000.$$

So, we shall proceed to develop expressions for the functions of the sum and difference of two angles in terms of the functions of the separate angles.

1. Functions of the sum of two angles. In Fig. 4–1 a circle of any radius ($OA = OF$) is constructed with the acute angles $x = FOD$ and $y = AOF$.

From the point A, a perpendicular is drawn to the horizontal axis at C and another to the radius OF at E. From the point E a perpendicular is drawn to the line AC at B.

Now, from geometry, the angle EAB is equal to the angle EOD (denoted by x) because the sides of the triangle EAB are drawn perpendicular to the respective sides of triangle EOD, thus making the triangles similar, which in turn makes their corresponding angles equal.

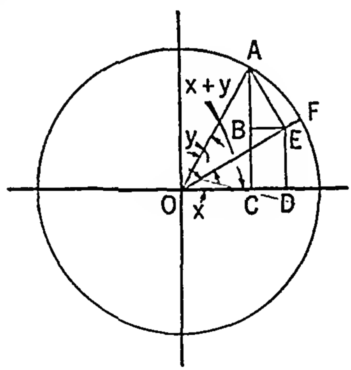


Fig. 4–1.

(a) $\sin (x + y)$:

From Figure 4–1 it is evident that $CB = DE$. Then

$$\sin AOC = \sin (x + y) = \frac{CA}{OA} = \frac{CB + BA}{OA} = \frac{DE + BA}{OA}.$$

But $$DE = (OE) \sin x$$
and $$BA = (AE) \cos x.$$

Thus, $$\sin (x + y) = \frac{OE \sin x + AE \cos x}{OA}$$

or $$\sin (x + y) = \frac{OE}{OA} \sin x + \frac{AE}{OA} \cos x.$$

But $$\frac{OE}{OA} = \cos y$$

and $$\frac{AE}{OA} = \sin y.$$

Therefore,

$$\sin (x + y) = \sin x \cos y + \cos x \sin y.$$

The above proof may be extended to include obtuse as well as acute angles and can be shown to apply for angles of any size.

So, it is plain that the expression $\sin (x + y)$ does not mean $\sin x + \sin y$ but means the sine of an angle which is represented as the sum of two other angles.

EXAMPLE 4–1. Find $\sin 75°$ when given $\sin 30° = 0.5$; $\cos 30° = 0.866$; and $\sin 45° = \cos 45° = 0.707$.

Solution: Let $x = 45°$ and $y = 30°$. Then

$$\sin (x + y) = \sin (45° + 30°),$$

$$\sin 75° = \sin (45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°,$$
$$\sin 75° = (0.707)(0.866) + (0.707)(0.5) = 0.612 + 0.354 = 0.966.$$

(b) $\cos (x + y)$:

From Fig. 4–1, the cosine of the sum of two angles may be determined. A study of the figure will show that

$$\cos (x + y) = \frac{OC}{OA} \text{ and } CD = BE.$$

Since $$OC = OD - CD \text{ or } OD - BE,$$

$$\cos (x + y) = \frac{OD - BE}{OA}.$$

Now $$OD = OE \cos x$$
and $$BE = AE \sin x.$$

Thus $$\cos (x + y) = \frac{OE \cos x - AE \sin x}{OA},$$

or $$\cos (x + y) = \frac{OE}{OA} \cos x - \frac{AE}{OA} \sin x.$$

But $$\frac{OE}{OA} = \cos y$$

and $$\frac{AE}{OA} = \sin y.$$

Therefore, $$\cos (x + y) = \cos x \cos y - \sin x \sin y.$$

This proof can also be extended to include all angles and is true under any and all conditions.

EXAMPLE 4–2. Find $\cos 75°$, when given the same values as in Example 4–1.

Solution: Let $x = 45°$ and $y = 30°$.

Then $\cos(x + y) = \cos(45° + 30°)$,

$$\cos(x + y) = \cos 45° \cos 30° - \sin 45° \sin 30°.$$

Therefore,
$$\cos 75° = (0.707)(0.866) - (0.707)(0.5)$$
$$= 0.612 - 0.354 = 0.258.$$

(c) $\tan(x + y)$:

From the two expressions just developed, it is a simple matter to determine the expressions for $\tan(x + y)$ and $\cot(x + y)$ since the tangent of an angle is equal to the sine divided by the cosine and the cotangent is equal to the cosine divided by the sine. Thus

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)},$$

and by substituting the values for $\sin(x + y)$ and $\cos(x + y)$,

$$\tan(x + y) = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

This is a rather cumbersome expression, and it is more convenient for use by putting it in terms of $\tan x$ and $\tan y$. In order to do this, we divide both numerator and denominator by $\cos x \cos y$. Then we have

$$\tan(x + y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

(d) $\cot(x + y)$:

In a similar way it can be shown that

$$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}.$$

EXAMPLE 4–3. Find $\tan 75°$ and $\cot 75°$, when given $\tan 30° = 0.577$; $\tan 45° = 1$; $\cot 30° = 1.732$; and $\cot 45° = 1$.

Solution: Let $x = 45°$ and $y = 30°$.

Then
$$\tan 75° = \tan(x + y) = \tan(45° + 30°),$$
$$\tan 75° = \frac{\tan 45° + \tan 30°}{1 - (\tan 45°)(\tan 30°)},$$
$$\tan 75° = \frac{1 + 0.577}{1 - (1)(0.577)} = \frac{1.577}{0.423} = 3.72.$$

Also,
$$\cot 75° = \cot(x + y) = \cot(45° + 30°),$$
$$\cot 75° = \frac{(\cot 45°)(\cot 30°) - 1}{\cot 45° + \cot 30°},$$
$$\cot 75° = \frac{(1)(1.732) - 1}{1 + 1.732} = \frac{0.732}{2.732} = 0.268.$$

EXAMPLE 4–4. Find tan 165° and cot 165° when given tan 30° = cot 60° = .577; tan 45° = cot 45° = 1; tan 60° = cot 30° = 1.732.

Solution: Tan 75° has been found in Example 4–3 to be 3.72 and cot 75° has been found to be .268.

$$\begin{aligned}
\tan 135^\circ &= \tan (75^\circ + 60^\circ) \\
&= \frac{\tan 75^\circ + \tan 60^\circ}{1 - (\tan 75^\circ)(\tan 60^\circ)} \\
&= \frac{(3.72 + 1.732)}{1 - (3.72)(1.732)} = \frac{5.45}{1 - 6.45} = \frac{5.45}{-5.45} = -1. \\
\tan 165^\circ &= \tan (135^\circ + 30^\circ) \\
&= \frac{\tan 135^\circ + \tan 30^\circ}{1 - (\tan 135^\circ)(\tan 30^\circ)} \\
&= \frac{-1 + 0.577}{1 - (-1)(0.577)} = \frac{-0.423}{1.577} = -0.268. \\
\cot 165^\circ &= \frac{1}{\tan 165^\circ} = \frac{1}{-0.268} = -3.732.
\end{aligned}$$

EXERCISE 4–1

1. Prove that the expression for sin $(x + y)$ holds true when $(x + y)$ is an obtuse angle, by reconstructing Fig. 4–1 to make $(x + y)$ obtuse.

2. Prove that the expression for cos $(x + y)$ holds true when $(x + y)$ is an obtuse angle, by reconstructing Fig. 4–1 to make $(x + y)$ obtuse.

Given

$$\sin 0^\circ = \cos 90^\circ = 0,$$
$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2},$$
$$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2},$$
$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2},$$

and

$$\sin 90^\circ = \cos 0^\circ = 1,$$

Determine the values in the following problems:

3. sin 90°. **4.** cos 90°. **5.** cos 15°. **6.** sin 15°. **7.** sin 105°. **8.** cos 105°. **9.** sin 135°. **10.** cos 135°. **11.** sin 165°. **12.** cos 165°.

13. Prove that $\cot (x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$.

Given

$$\tan 30^\circ = \cot 60^\circ = \frac{\sqrt{3}}{3},$$
$$\tan 45^\circ = \cot 45^\circ = 1,$$

and

$$\tan 60^\circ = \cot 30^\circ = \sqrt{3},$$

Determine the values in the following problems:

14. tan 15°. **17.** cot 90°. **20.** tan 135°. **23.** cot 165°.
15. cot 15°. **18.** tan 105°. **21.** cot 135°.
16. tan 90°. **19.** cot 105°. **22.** tan 165°.

2. Functions of the difference of two angles. In Fig. 4–2 there is constructed a circle of any radius ($OA = OB$) with the acute angles $AOB = x$ and $AOE = y$. This makes angle $BOE = x - y$. Two perpendiculars are erected from point E, ED perpendicular to OD and EG perpendicular to OA. Also, two perpendiculars are erected from the point

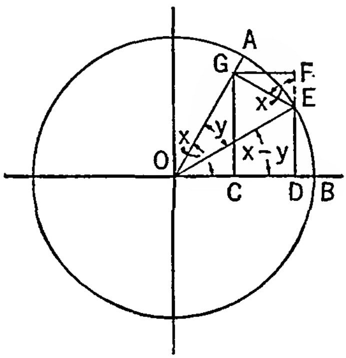


Fig. 4–2.

G, GC perpendicular to OD and GF perpendicular to ED produced to point F. Then, by construction, the triangle GEF is similar to triangle GOC since its sides are perpendicular to the sides of triangle GOC. This then makes angle GEF = angle x.

(a) $\sin(x - y)$:

From Fig. 4–2

$$\sin(x - y) = \frac{DE}{OE} = \frac{FD - FE}{OE} = \frac{GC - FE}{OE} = \frac{GC}{OE} - \frac{FE}{OE}.$$

But $\qquad GC = OG \sin x$

and $\qquad FE = GE \cos x.$

Thus, $\sin(x - y) = \dfrac{OG \sin x}{OE} - \dfrac{GE \cos x}{OE}.$

Also, $\qquad \dfrac{OG}{OE} = \cos y$

and $\qquad \dfrac{GE}{OE} = \sin y.$

Therefore, $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

As in the case of $\sin(x + y)$, this proof can be extended to include angles of any size.

EXAMPLE 4–5. Find sin 15° when given $\sin 45° = \cos 45° = 0.707$; $\sin 30° = 0.5$; and $\cos 30° = 0.866$.

Solution: Let $x = 45°$ and $y = 30°$.

Then $\quad \sin 15° = \sin (45° - 30°)$.
Therefore, $\quad \sin 15° = \sin 45° \cos 30° - \cos 45° \sin 30°$

$$\sin 15° = (0.707)(0.866) - (0.707)(0.5) = 0.6123 - 0.3535 = 0.2588.$$

(b) $\cos (x - y)$:

From Fig. 4–2 the cosine of the difference of two angles also may be determined.

$$\cos (x - y) = \frac{OD}{OE} = \frac{OC + CD}{OE} = \frac{OC + GF}{OE} = \frac{OC}{OE} + \frac{GF}{OE}.$$

But $\quad OC = OG \cos x$
and $\quad GF = GE \sin x$.

Thus $\quad \cos (x - y) = \dfrac{OG \cos x}{OE} + \dfrac{GE \sin x}{OE}$.

Also $\quad \dfrac{OG}{OE} = \cos y$

and $\quad \dfrac{GE}{OE} = \sin y$.

Therefore,

$$\cos (x - y) = \cos x \cos y + \sin x \sin y.$$

EXAMPLE 4–6. Find cos 15° from the values given in Example 4–5.

Solution:

$$\cos 15° = \cos (45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$\cos 15° = (0.707)(0.866) + (0.707)(0.5) = 0.6123 + 0.3535 = 0.9658.$$

(c) $\tan (x - y)$ and $\cot (x - y)$:

From the expressions just developed for $\sin (x - y)$ and $\cos (x - y)$, it is a simple matter to derive other expressions giving $\tan (x - y)$ and $\cot (x - y)$, since the tangent of an angle is equal to the sine divided by the cosine and the cotangent of an angle is equal to the cosine divided by the sine. It will then be found that

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

and

$$\cot (x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

EXAMPLE 4–7. Find tan 15° from the data given in Example 4–5.

Solution:

$$\tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°},$$

$$\tan 45° = \frac{\sin 45°}{\cos 45°} = \frac{0.707}{0.707} = 1,$$

and $$\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{0.500}{0.866} = 0.577.$$

Therefore, $$\tan 15° = \frac{1 - 0.577}{1 + (1)(0.577)} = \frac{0.423}{1.577} = 0.268.$$

3. Applied problems. The expressions that have been developed in the preceding sections may be summarized thus:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}.$$

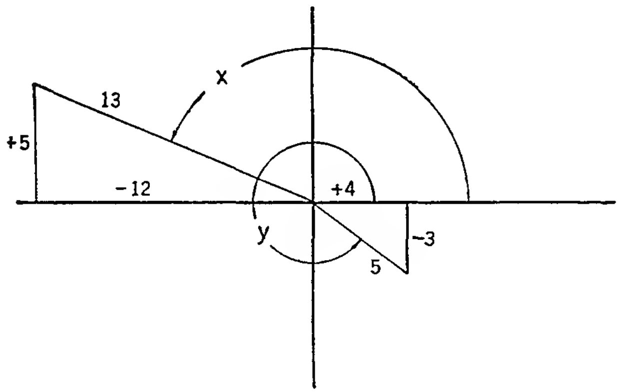


Fig. 4–3.

An examination of these summarized expressions will show that the signs must be taken in the order in which they are written; that is, *the top sign on the left-hand side is used with the top sign on the right-hand side.*

EXAMPLE 4–8. Find the functions of the angle $(x + y)$, given

$$x_2 = \cos^{-1} - \frac{12}{13} \text{ and } y_4 = \tan^{-1} - \frac{3}{4}.$$

Solution: Since the subscripts indicate the respective quadrants in which the angles lie, it is apparent that angle x is in the second quadrant

and angle y is in the fourth quadrant. Fig. 4–3 shows the angles in the proper quadrants.

$$\sin x = +\frac{5}{13} \qquad \sin y = -\frac{3}{5}$$
$$\cos x = -\frac{12}{13} \qquad \cos y = +\frac{4}{5}$$
$$\tan x = -\frac{5}{12} \qquad \tan y = -\frac{3}{4}$$
$$\cot x = -\frac{12}{5} \qquad \cot y = -\frac{4}{3}$$

The values of the functions of the angles $(x + y)$ and $(x - y)$ can be determined easily.

$$\sin (x + y) = \sin x \cos y + \cos x \sin y,$$
$$\sin (x + y) = \left(+\frac{5}{13}\right)\left(+\frac{4}{5}\right) + \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right),$$
$$\sin (x + y) = \frac{20}{65} + \frac{36}{65} = +\frac{56}{65}.$$

Then, as an inverse function, $(x + y) = \sin^{-1}\left(+\frac{56}{65}\right)$.

$$\sin (x - y) = \sin x \cos y - \cos x \sin y,$$
$$\sin (x - y) = \left(+\frac{5}{13}\right)\left(+\frac{4}{5}\right) - \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right),$$
$$\sin (x - y) = \frac{20}{65} - \frac{36}{65} = -\frac{16}{65}.$$

As an inverse function, $(x - y) = \sin^{-1}\left(-\frac{16}{65}\right)$.

EXERCISE 4–2

1. Show that $\tan (x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$.
2. Show that $\cot (x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$.
3. Check the values of each of the functions of 45° by using the values of the functions of 75° and 30° in the expressions for the difference of two angles.
4. Determine the other functions of $(x + y)$ in Example 4–8.
5. Determine the other functions of $(x - y)$ in Example 4–8.
6. If $\sin A_2 = \frac{4}{5}$ and $\cos B_1 = \frac{5}{13}$, find the functions of $(A + B)$.
7. In Problem 6 find the functions of $(A - B)$.

8. If $A_2 = \tan^{-1}\left(-\frac{8}{15}\right)$ and $B_3 = \cos^{-1}\left(-\frac{4}{5}\right)$, find the functions of $(A + B)$.

9. In Problem 6, find the functions of $(B - A)$.

4. Functions of twice an angle. By substituting x for y in the expressions for the functions of $(x + y)$, it is possible to derive formulas for the functions of twice an angle.

Since $\sin(x + y) = \sin x \cos y + \cos x \sin y$, the substitution of x for y will give $\sin(x + x) = \sin x \cos x + \cos x \sin x$, or

$$\sin 2x = 2 \sin x \cos x.$$

Similarly,

$$\cos 2x = \cos^2 x - \sin^2 x,$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x},$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}.$$

In all the above formulas, the angles on the left-hand side of the equations are two times those on the right-hand side.

5. Functions of half an angle. By substituting $\frac{x}{2}$ for the angle x in the identities,

$$\sin^2 x + \cos^2 x = 1 \quad \text{and} \quad \cos^2 x - \sin^2 x = \cos 2x,$$

there results

$$(1)\ \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1$$

$$(2)\ \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos x.$$

Subtracting (2) from (1),

$$2 \sin^2 \frac{x}{2} = 1 - \cos x,$$

or

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}.$$

Then

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}.$$

By adding (1) and (2), there results

$$2 \cos^2 \frac{x}{2} = 1 + \cos x,$$

whence

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}.$$

Since

$$\tan = \frac{\sin}{\cos} \text{ and } \cot = \frac{\cos}{\sin},$$

then $$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

and $$\cot\frac{x}{2} = \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} = \pm\sqrt{\frac{1+\cos x}{1-\cos x}}.$$

In all the above formulas, the angles on the left-hand side of the equation are just half those on the right-hand side and *angles of any value may be used.* The sign before the radical will be determined by the quadrant in which the half angle is located.

EXERCISE 4-3

1. As indicated above, develop the expression for $\cos 2x$ in terms of angle x.
2. Develop the expression for $\tan 2x$ in terms of $\tan x$.
3. Develop the expression for $\cot 2x$ in terms of $\cot x$.
4. Develop an expression for $\sin 3x$ in terms of $\sin x$.
5. Develop an expression for $\cos 3x$ in terms of $\cos x$.

[*Hint:* In Problem 4, let $y = 2x$ in the expression $\sin (x + y)$ thereby getting $\sin (x + 2x)$, which gives $\sin 3x$. Also, in Problem 5, let $\cos 3x = \cos (x + 2x)$.]

6. If $\sin x = \frac{\sqrt{3}}{2}$ and $\cos x = \frac{1}{2}$ find $\sin 2x$.
7. Find $\cos 2x$ for the angle of Problem 6.
8. Find $\tan 2x$ for the angle of Problem 6.
9. Find $\cot 2x$ for the angle of Problem 6.
10. Determine the functions of 15°, given $\sin 30° = \frac{1}{2}$.
11. If $\sin x = 0.6$, determine the functions of $\frac{x}{2}$.
12. If $\sin x = 0.8$, determine the functions of $\frac{x}{2}$.

6. Sums and differences of functions. Now that we know

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

and $$\sin (x - y) = \sin x \cos y - \cos x \sin y,$$

we can either add these two equations or subtract them and obtain the following:

$$\sin (x + y) + \sin (x - y) = 2 \sin x \cos y$$

and $$\sin (x + y) - \sin (x - y) = 2 \cos x \sin y.$$

In a similar way by using the formulas for $\cos (x \pm y)$, there results:

$$\cos (x + y) + \cos (x - y) = 2 \cos x \cos y$$

and $$\cos (x + y) - \cos (x - y) = -2 \sin x \sin y.$$

By letting $x + y = A$ and $x - y = B$, we have:

By addition

$$\begin{array}{rl} x + y &= A \\ x - y &= B \\ \hline 2x &= A + B \\ x &= \dfrac{A + B}{2} \end{array}$$

By subtraction

$$\begin{array}{rl} x + y &= A \\ x - y &= B \\ - \quad + & \quad - \\ \hline 2y &= A - B \\ y &= \dfrac{A - B}{2} \end{array}$$

Then our previous equations may be written as follows:

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}.$$

$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}.$$

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}.$$

$$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}.$$

Dividing the sum of the sines by the difference of the sines, there results

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2 \sin \dfrac{A + B}{2} \cos \dfrac{A - B}{2}}{2 \cos \dfrac{A + B}{2} \sin \dfrac{A - B}{2}},$$

or $$\frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{A + B}{2} \cot \frac{A - B}{2}.$$

But $$\cot \frac{A - B}{2} = \frac{1}{\tan \dfrac{A - B}{2}}.$$

Therefore, $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \dfrac{A + B}{2}}{\tan \dfrac{A - B}{2}}.$$

This last expression is important since it is used in the development of the law of tangents.

The sums or differences of the functions of angles are useful in breaking down certain types of identities into simpler forms. The following example illustrates the procedure:

EXAMPLE 4–9. Show that

$$\frac{\cos 3x - \cos x}{\cos 3x + \cos x} = -\tan 2x \tan x.$$

Solution:

$$\cos 3x - \cos x = -2 \sin \frac{3x + x}{2} \sin \frac{3x - x}{2}$$
$$= -2 \sin 2x \sin x,$$
$$\cos 3x + \cos x = 2 \cos \frac{3x + x}{2} \cos \frac{3x - x}{2}$$
$$= 2 \cos 2x \cos x.$$

Therefore,

$$\frac{\cos 3x - \cos x}{\cos 3x + \cos x} = \frac{-2 \sin 2x \sin x}{2 \cos 2x \cos x}$$
$$= -\tan 2x \tan x.$$

EXERCISE 4–4

Prove the following:

1. $\dfrac{\sin 5x - \sin 3x}{\sin 5x + \sin 3x} = \tan x \cot 4x.$
2. $\dfrac{\cos 3x - \cos x}{\sin 3x - \sin x} = \dfrac{2 \tan x}{\tan^2 x - 1}.$
3. $\dfrac{\cos 3x + \cos x}{\sin 3x + \sin x} = \cot 2x.$
4. $\dfrac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x} = \tan x.$

EXERCISE 4–5

1. If $\cos (A + B) = \frac{3}{4}$ and $\sin B = \frac{1}{3}$, find the functions of angle A, assuming all angles to be acute.

2. In a three-phase electrical circuit, the power may be measured by the use of two single-phase wattmeters, and it can be shown that the equations for the readings of the wattmeters are

$$W_1 = VI \cos (30° - \theta),$$
$$W_2 = VI \cos (30° + \theta).$$

Show that the total power $(W_1 + W_2)$ is equal to $\sqrt{3}\, VI \cos \theta$.

3. From Problem 2, show that

$$\tan \theta = \sqrt{3}\, \frac{W_1 - W_2}{W_1 + W_2}.$$

4. The effective value of an alternating-current wave is defined as that value that will give the same heating effect as the same value of direct current, and heating effect is proportional to the square of the current.

From the equation,

$$i^2 = I_m^2 \sin^2 \theta,$$

derive the following squared-current wave equation:

$$i^2 = I_m^2 \left(\frac{1 - \cos 2\theta}{2}\right).$$

5. Two voltages in an alternating current circuit are defined by the following expressions:

$$e_1 = 120 \sin (x - 30°),$$
$$e_2 = 100 \sin (x - 90°).$$

Find the sum of these two voltages.

6. The range of a projectile is given by the equation

$$R = \frac{V^2 \sin 2\theta}{32}.$$

Convert into an equation containing functions of the angle θ.

REVIEW EXERCISE 4-6

Simplify the following:

1. $\dfrac{\sin 7x - \sin 5x}{\cos 7x - \cos 5x}.$
2. $\dfrac{\cos 5x - \cos x}{\sin 5x + \sin x}.$
3. $\dfrac{\sin 5x - \sin x}{\cos 5x + \cos x}.$
4. $\dfrac{\cos 2x - \cos 2y}{2(\cos x + \cos y)}.$
5. $(\tan 2x - \tan 3x)(\cos 7x + \cos x).$
6. $\left(\dfrac{\sin 3x + \sin x}{\sin 3x - \sin x}\right)\left(\dfrac{\cos 3x + \cos x}{\cos 3x - \cos x}\right).$
7. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x}.$
8. $\dfrac{\sin (A + B) \sin (A - B)}{\cos^2 A \cos^2 B}.$
9. $\dfrac{\cos 2A - \sin 2A + 1}{\sin 2A + \cos 2A - 1}.$
10. $\dfrac{\cos A + \sin A}{\cos A - \sin A} - \dfrac{\sin 3A + \sin A}{\cos 3A + \cos A}.$

If $A_2 = \sin^{-1} \dfrac{8}{17}$ and $B_4 = \cos^{-1} \dfrac{4}{5}$, find:

11. $\cos(A+B)$. **13.** $\sin(A+B)$. **15.** $\tan(A+B)$.
12. $\cos(A-B)$. **14.** $\sin(A-B)$. **16.** $\tan(A-B)$.

If $A_3 = \cos^{-1} - \frac{3}{5}$, find:

17. $\cos 2A$. **19.** $\sin 2A$. **21.** $\tan 2A$.
18. $\cos \frac{A}{2}$. **20.** $\sin \frac{A}{2}$. **22.** $\tan \frac{A}{2}$.

If $A_2 = \cos^{-1} - \frac{12}{13}$ and $B_3 = \sin^{-1} - \frac{3}{5}$, find:

23. $\sin(A+B)$. **25.** $\tan(A-B)$. **27.** $\cos(A+B)$.
24. $\cos(A-B)$. **26.** $\sin(A-B)$. **28.** $\tan(A+B)$.

If $A_3 = \sin^{-1} - \frac{4}{5}$, find:

29. $\sin 2A$. **31.** $\cos \frac{A}{2}$. **33.** $\tan 2A$.
30. $\tan \frac{A}{2}$. **32.** $\sin \frac{A}{2}$. **34.** $\cos 2A$.

If $A_3 = \tan^{-1} \frac{3}{4}$ and $B_3 = \cos^{-1} - \frac{5}{13}$, find:

35. $\cos(A-B)$. **37.** $\sin(A+B)$. **39.** $\tan(A-B)$.
36. $\tan(A+B)$. **38.** $\cos(A+B)$. **40.** $\sin(A-B)$.

Chapter 5

THE SINE AND COSINE LAWS WITH APPLICATIONS

THE SOLUTION of the oblique triangle by the use of right-triangle methods was considered in Chapter 2. This consisted of dividing the oblique triangle into two right triangles and solving these right triangles.

Now, it is a comparatively easy matter to develop general expressions that will apply to all oblique triangles and enable us to make use of a table of logarithms or a slide rule in the solution of a triangle.

Such a solution consists in determining all unknown parts of the triangle. With six parts of a triangle, three sides and three angles, at least three parts, one of which should be a side, must be given in order to find the other parts.

One general expression with which we are already familiar is that the sum of the angles of a triangle is equal to 180°. This will enable us to find one angle of a triangle if the other two angles are given.

The method used in the solution of a triangle will depend upon which set of three parts is given. There are four possible cases. There may be given

(a) a side and two angles,
(b) two sides and the angle opposite one of them,
(c) two sides and the angle between them,
(d) three sides.

1. The sine law. The sine law states that *any two sides of a triangle are in the same ratio as the sines of the respectively opposite angles.*

Consider the oblique triangle ABC in each of the following figures:

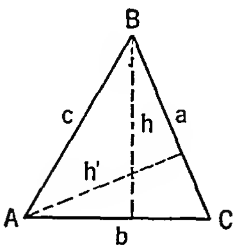


Fig. 5–1.

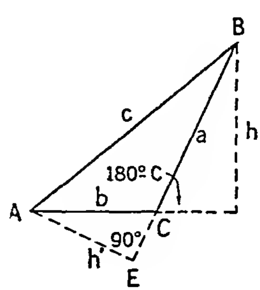


Fig. 5–2.

In these figures the following relations can be seen:

	Fig. 5–1	Fig. 5–2
	$\dfrac{h}{c} = \sin A$	$\dfrac{h}{c} = \sin A$
or	$h = c \sin A,$	$h = c \sin A,$
also	$\dfrac{h}{a} = \sin C$	$\dfrac{h}{a} = \sin (180 - C) = \sin C$
or	$h = a \sin C.$	$h = a \sin C.$

Therefore,

$$c \sin A = a \sin C$$

or

$$\frac{a}{c} = \frac{\sin A}{\sin C}.$$

Therefore,

$$c \sin A = a \sin C \tag{1}$$

or

$$\frac{a}{c} = \frac{\sin A}{\sin C}.$$

By erecting the perpendicular h' to the side BC or BC extended, it can be shown in the same way as above that

$$\frac{b}{c} = \frac{\sin B}{\sin C}. \tag{2}$$

Combining expressions (1) and (2), it is evident that the following relations exist:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

An examination of these formulas will show that a triangle may be solved by use of the law of sines when two angles and a side or two sides and the angle opposite one of them are given. It is also evident that the sine law is workable with a table of logarithms or with the slide rule.

The examples that follow illustrate the use of the sine law.

Example 5–1. Given a side and two angles. Given $a = 24.3$, $A = 46° 18'$, and $B = 23° 14'$ in the oblique triangle ABC, solve for the unknown parts.

Solution: A systematic arrangement will aid in the solution.

$$\text{Angle } C = 180° - (A + B) = 180° - 69° 32' = 110° 28'.$$

$$\frac{a}{b} = \frac{\sin A}{\sin B},$$

or

$$b = \frac{a \sin B}{\sin A} = \frac{(24.3) \sin 23° 14'}{\sin 46° 18'} = \frac{(24.3)(0.3945)}{0.723},$$

$$b = 13.26;$$

also

$$\frac{a}{c} = \frac{\sin A}{\sin C},$$

or

$$c = \frac{a \sin C}{\sin A} = \frac{(24.3)(\sin 110° 28')}{\sin 46° 18'} = \frac{(24.3)(0.937)}{0.723},$$

$$c = 31.5.$$

The foregoing solution has been worked out with a slide rule. An alternate solution by the use of logarithms follows:

Since $$b = \frac{a \sin B}{\sin A} = \frac{(24.3)(\sin 23° 14')}{\sin 46° 18'},$$

then $$\log b = \log 24.3 + \log \sin 23° 14' - \log \sin 46° 18',$$

$$\begin{aligned} \log 24.3 &= \quad 1.3856 \\ \log \sin 23° 14' &= \quad \underline{9.5960 - 10} \\ & \quad\ \ 10.9816 - 10 \\ \log \sin 46° 18' &= -\ \underline{9.8591 - 10} \\ \log b &= \quad 1.1225, \\ b &= \quad 13.259. \end{aligned}$$

Since $$c = \frac{a \sin C}{\sin A} = \frac{(24.3)(\sin 110° 28')}{\sin 46° 18'},$$

then $$\log c = \log 24.3 + \log \sin 110° 28' - \log \sin 46° 18',$$

$$\begin{aligned} \log 24.3 &= \quad 1.3856 \\ \log \sin 110° 28' &= \quad \underline{9.9717 - 10} \\ & \quad\ \ 11.3573 - 10 \\ \log \sin 46° 18' &= -\ \underline{9.8591 \mp 10} \\ \log c &= \quad 1.4982, \\ c &= \quad 31.49. \end{aligned}$$

An inspection of these two solutions will show that slide-rule calculation should be satisfactory for most problems.

EXERCISE 5–1

1. Develop the law of sines from Figs. 5–1 and 5–2 by erecting the perpendicular to side BC; also by erecting the perpendicular to side AB.

Solve the triangle ABC in each of the following problems, given the parts indicated:

2. $a = 500$, $A = 10° 11'$, $B = 46° 34'$.
3. $b = 13.6$, $B = 13° 50'$, $C = 57° 15'$.
4. $c = 1005$, $A = 78° 15'$, $B = 54° 37'$.
5. $a = 805$, $A = 99° 50'$, $B = 45° 45'$.
6. $b = 990$, $A = 37° 52'$, $C = 65° 5'$.
7. $a = 820$, $A = 12° 50'$, $B = 140° 58'$.

8. The angles at the base of a triangle are 30° and 120° and the base is 600 ft long. Find the other sides and the altitude.

9. In a trapezoid the parallel sides are equal to m and n, respectively, and the angles at the ends of one of the parallel sides are A and B. Find the nonparallel sides.

10. Compute the results in Problem 9 when $m = 14$, $n = 8$, $A = 65°$, $B = 42°$.

11. In a parallelogram the diagonal d is given and the angles θ and Φ which this diagonal makes with the sides. Find the sides.

12. Compute the results in Problem 11 when $d = 12.3$, $\theta = 19° 10'$, and $\Phi = 41° 55'$.

13. Two voltages, one of 100 and the other unknown, make an angle of 48° 10′ with each other and together give a resultant voltage of 150. Find the unknown voltage.

14. It is desired to find the width of a stream. The points A and B on opposite banks and another point C on the same bank as A are chosen. C is found to be 990 ft from A. The angles ABC and ACB are measured and found to be 33° 25′ and 47° 30′, respectively. Find AB.

2. The ambiguous case. The sine law may be used also for the solution of an oblique triangle when given two sides and the angle opposite one of them. This, however, means solving for the sine of the angle opposite the second side and, since there may be two angles for any positive sine value, the solution leads to what is termed the *ambiguous case.* The name results from the fact that the problem may have one solution, two solutions, or no solution at all. Let us consider a case in which the sides a and c and the acute angle A are given. We can draw the side $AB = c$ at the angle A with the horizontal as shown in Fig. 5-3.

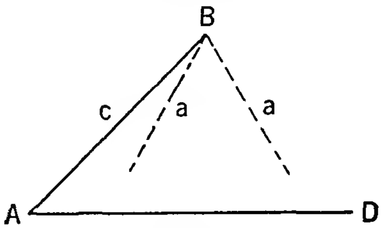


Fig. 5-3.

Now the vertex C of the triangle is not known, and in order to find it we must swing an arc from the point B with a radius equal to the given side a. The vertex C then will be found where this arc meets the line AD.

From this construction, it is evident that there are several possibilities.

(a) If the side a is greater than the perpendicular distance from the vertex B to AD but is smaller than c, then the arc from B intersects the line AD in two places, C^1 and C^2, each of which is the vertex of a triangle that will satisfy the given conditions. Therefore, for a complete solution, it is necessary to solve both triangles ABC^1 and ABC^2, thus giving two solutions. In this case a is less than c (written $a < c$) but is greater than $c \sin A$ (written $a > c \sin A$) and is illustrated in Fig. 5-4.

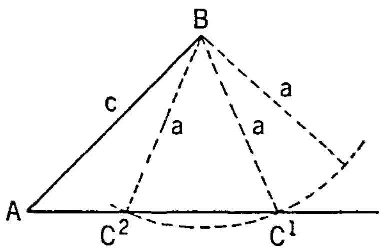


Fig. 5-4.

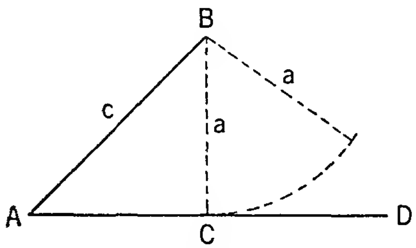


Fig. 5-5.

(b) If the side a is just equal to the perpendicular distance from the vertex B to AD, then there is only one solution, since the point C will lie

at the foot of this perpendicular and can occupy no other position. The triangle ABC will be a right triangle and $\sin C = 1$. Therefore, a will be just equal to $c \sin A$. This condition for one and only one solution is illustrated in Fig. 5–5.

(c) If the side a is smaller than the length of the perpendicular distance from the vertex B to AD, the arc from B will not meet the line AD at any point and there will be no solution for the problem. In this case a is less than $c \sin A$ (written $a < c \sin A$) and is illustrated in Fig. 5–6.

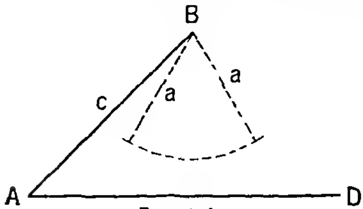


Fig. 5–6.

(d) There remains but one other possibility. This occurs when a is equal to or larger than c. If side a is equal to side c, only one triangle will be formed by swinging an arc from point B and the triangle will be isosceles. However, if a is larger than c, an arc swung from point B will cut the line AD in two points C^1 and C^2 on opposite sides of the point A. In the two triangles thus formed, ABC^1 and ABC^2, one contains the given angle A while the other contains the supplementary angle for A, or $180° - A$. Therefore, again there will be but one solution. The conditions are illustrated in Figs. 5–7 and 5–8.

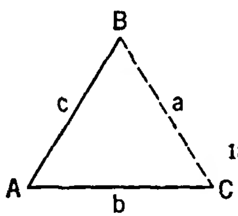


Fig. 5–7.

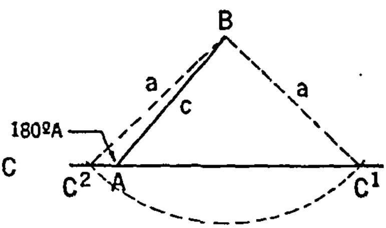


Fig. 5–8.

The conditions in the ambiguous case can be summarized in the following rules:

1. If a is less than c but greater than c sin A and angle A is acute, there are two solutions.

2. If a = c sin A and angle A is acute, there is only one solution and the triangle is a right triangle.

3. If a is equal to c, there is one solution and the triangle is isosceles.

4. If a is larger than c, there is one solution.

5. If a is smaller than c sin A, there is no solution.

It should be kept in mind that the given conditions may involve any two sides and the angle opposite one of them. Instead of sides a and c and angle A as shown in the above rules, sides b and c and angle B, or sides a and b and angle B might have been given. Therefore, the letters used in

the rules must be made to agree with the conditions given in the problem. The following list shows the possible conditions:

Given sides		Given angle
a	c	A
a	b	A
a	b	B
b	c	B
b	c	C
a	c	C

EXAMPLE 5–2. Given a triangle in which $a = 6.5$, $c = 8.5$, and angle $A = 30°$, solve for the other parts.

Solution: Here a is less than c and angle A is acute. Therefore, we must investigate the value of $c \sin A$ to determine whether there is a solution and, if so, how many solutions.

$$c \sin A = (8.5)(\sin 30°) = (8.5)\left(\frac{1}{2}\right) = 4.25.$$

Since a is smaller than c but larger than $c \sin A$, two solutions are possible.

$$\sin C = \frac{c \sin A}{a} = \frac{(8.5)(0.5)}{6.5} = 0.654.$$

Then $\quad C = 40° 50' \quad \text{or} \quad 139° 10'$

and $\quad B = 180° - (40° 50' + 30°) \quad \text{or} \quad 180° - (139° 10' + 30°),$

$$B = 109° 10' \quad \text{or} \quad 10° 50';$$

$$b = \frac{a \sin B}{\sin A},$$

$$b = \frac{(6.5) \sin 109° 10'}{\sin 30°} \quad \text{or} \quad b = \frac{(6.5) \sin 10° 50'}{\sin 30°},$$

$$b = \frac{(6.5)(0.9446)}{0.5} \quad \text{or} \quad b = \frac{(6.5)(0.188)}{0.5},$$

$$b = 12.28 \quad \text{or} \quad b = 2.44.$$

The two triangles that satisfy the given conditions are then as follows:

I	II
$a = 6.5$	$a = 6.5$
$b = 12.28$	$b = 2.44$
$c = 8.5$	$c = 8.5$
$A = 30°$	$A = 30°$
$B = 109° 10'$	$B = 10° 50'$
$C = 40° 50'$	$C = 139° 10'$

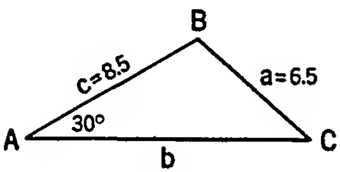


Fig. 5–9.

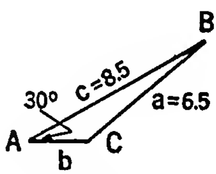


Fig. 5–10.

EXAMPLE 5-3. Solve the triangle when $a = 8$, $c = 16$, and $A = 30°$.

Solution:

$$c \sin A = 16 \sin 30° = (16)\left(\frac{1}{2}\right) = 8.$$

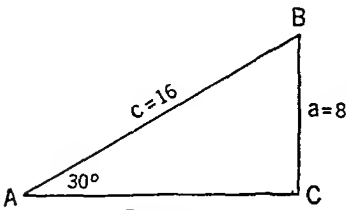


Fig. 5-11.

Here a is less than c but is equal to $c \sin A$. Therefore, there is but one solution and the triangle is a right triangle.

Then $C = 90°$ and $B = 60°$,

$$\text{and } b = \frac{a \sin B}{\sin A} = \frac{(8)(\sin 60°)}{\sin 30°} = \frac{(8)\frac{\sqrt{3}}{2}}{\frac{1}{2}},$$

$$b = 8\sqrt{3} = 13.856.$$

EXAMPLE 5-4. Solve the triangle when $a = 2$, $c = 8$, and $A = 30°$.

Solution: $$c \sin A = (8)\left(\frac{1}{2}\right) = 4.$$

In this case a is smaller than the value of $c \sin A$, and therefore there is no solution. See Fig. 5-12.

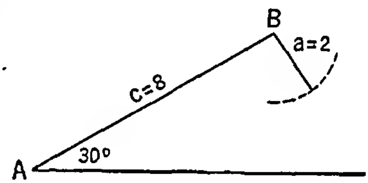


Fig. 5-12.

EXAMPLE 5-5. Solve the triangle when $a = 10$, $c = 8$, and $A = 30°$.

Solution: Here a is larger than c; hence there is but one solution, and angle A is larger than angle c because the larger angle is opposite the larger side.

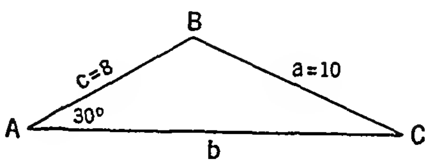


Fig. 5-13.

$$\sin C = \frac{c \sin A}{a} = \frac{(8)(\sin 30°)}{10} = \frac{(8)\left(\frac{1}{2}\right)}{10} = 0.4,$$

$$C = 23° 35'.$$

$$B = 180° - (A + C) = 180° - (23° 35' + 30°) = 126° 25'.$$

$$b = \frac{a \sin B}{\sin A} = \frac{(10)(\sin 126° 25')}{\sin 30°}$$

$$= \frac{(10)(0.8047)}{\frac{1}{2}} = 16.094.$$

3. Extending the sine law. There is an extension of the law of sines in connection with a circle circumscribed* about a triangle that is sometimes useful.

Circumscribe a circle about the triangle ABC as shown in Fig. 5–14, and denote the radii OA and OC by R. Erect the perpendicular ON to

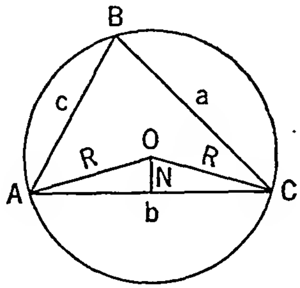


Fig. 5–14.

AC. Now, the angle AOC is a central angle intercepting the same arc as angle ABC and therefore by geometry,

$$\text{angle } AOC = 2B$$

or

$$\text{angle } AON = B$$

(an inscribed angle is measured by half the intercepted arc).

Thus $$AN = R \sin AON = R \sin B.$$

But $$AN = \frac{b}{2}.$$

Therefore, $$\frac{b}{2} = R \sin B \quad \text{or} \quad b = 2R \sin B.$$

In like manner it can be shown that

$$a = 2R \sin A$$

and

$$c = 2R \sin C.$$

Therefore, $$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

That is, *the ratio of any side of a triangle to the sine of the opposite angle is equal numerically to the diameter of the circle circumscribed about the triangle.*

Another interesting extension of the sine law is concerned with the perimeter (P) of a triangle. The equation can be developed readily.

The law of sines may be written in any one of the three forms:

$$\frac{a}{b} = \frac{\sin A}{\sin B}; \quad \frac{b}{c} = \frac{\sin B}{\sin C}; \quad \frac{a}{c} = \frac{\sin A}{\sin C}.$$

---

* Center of circumscribed circle is the intersection of perpendicular bisectors of sides of the triangle.

Then, by composition:

$$\frac{a}{a+b} = \frac{\sin A}{\sin A + \sin B},$$

or

$$\frac{a}{\sin A} = \frac{a+b}{\sin A + \sin B}.$$

Since

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

then:

$$\frac{a+b}{\sin A + \sin B} = \frac{c}{\sin C} \text{ or } \frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}.$$

Again by composition:

$$\frac{a+b+c}{c} = \frac{\sin A + \sin B + \sin C}{\sin C},$$

or

$$\frac{a+b+c}{\sin A + \sin B + \sin C} = \frac{c}{\sin C}.$$

Now, since $a + b + c = P$,

then:

$$\frac{P}{\text{sum of sines}} = \frac{c}{\sin C} = \frac{b}{\sin B} = \frac{a}{\sin A}.$$

That is, the perimeter of a triangle is in the same ratio to the sum of the sines of the angles as any side is to the sine of its opposite angle.

EXERCISE 5-2

In the following problems, determine the number of solutions in each and complete the solution for all possible triangles:

1. $a = 8.6$, $c = 9.72$, $A = 54° 40'$.
2. $a = 72.6$, $b = 117.5$, $A = 80° 3'$.
3. $b = 4.4$, $c = 5.21$, $B = 57° 37'.3$.
4. $a = 24.5$, $b = 19.8$, $A = 67° 32'$.
5. $b = 4.29$, $c = 2.01$, $C = 29° 52'.5$.
6. $a = 0.438$, $b = 0.558$, $B = 41° 31'$.
7. $a = 75$, $b = 29$, $B = 16° 15'$.

8. In the triangle ABC, $A = 75°$, $B = 70°$, and $c = 5$ in. Find the diameter of the circumscribed circle.

9. The diameter of a circle circumscribed about a triangle is 10 in. The angles A and B are equal to $76° 37'$ and $81° 46'$, respectively. Solve the triangle for the sides.

4. The cosine law. We have found that any oblique triangle may be solved by means of the sine law, provided a side, and two angles or two sides and the angle opposite one of them are given. There remain, however, the possibilities of being given either two sides and the angle included

between them or three sides. Neither of these latter can be solved by the use of the sine law and therefore there has been developed for such cases an expression called the *cosine law.*

The cosine law states that *the square of any side of a triangle is equal to the sum of the squares of the other two sides minus two times their product multiplied by the cosine of the angle between them.*

Consider the triangle ABC in either Fig. 5–15 or Fig. 5–16, in which any sides such as b and c and the angle θ between them are given. In Fig. 5–15, the angle θ is chosen as an acute angle, whereas in Fig. 5–16, it is chosen as an obtuse angle.

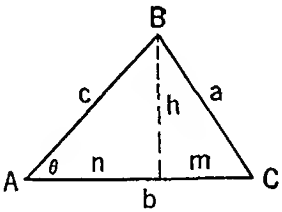


Fig. 5–15.

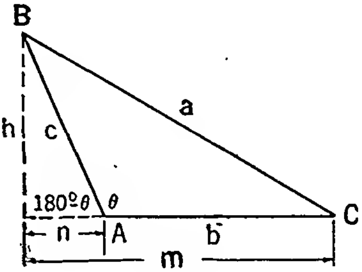


Fig. 5–16.

In Fig. 5–15:

$$\begin{aligned} a^2 &= h^2 + m^2 \\ &= h^2 + (b - n)^2 \\ &= h^2 + b^2 - 2bn + n^2. \end{aligned}$$

But $h^2 + n^2 = c^2$

and $\qquad n = c \cos \theta.$

Therefore, by substitution,

$$a^2 = b^2 + c^2 - 2bc \cos \theta.$$

In Fig. 5–16:

$$\begin{aligned} a^2 &= h^2 + m^2 \\ &= h^2 + (b + n)^2 \\ &= h^2 + b^2 + 2bn + n^2. \end{aligned}$$

But $h^2 + n^2 = c^2$

and $\qquad n = c \cos (180° - \theta)$
$\qquad\qquad = -c \cos \theta.$

Therefore,

$$\begin{aligned} a^2 &= b^2 + c^2 + 2b(-c \cos \theta) \\ &= b^2 + c^2 - 2bc \cos \theta. \end{aligned}$$

The proof of the cosine law would be the same for any two given sides and the included angle. The three forms in which it is written are as follows:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A, \\ b^2 &= a^2 + c^2 - 2ac \cos B, \\ c^2 &= a^2 + b^2 - 2ab \cos C. \end{aligned}$$

It will be evident at a glance that these expressions are not adapted to logarithmic computation, since the addition and subtraction of numbers is not done by means of logarithms. Therefore, the cosine law is usually employed where the given sides are easily squared.

A further inspection will show that if the included angle is a right angle, the third term will drop out of the expression since the cosine of 90° is zero and it will then become the familiar expression for the right triangle;

the square on the hypotenuse is equal to the sum of the squares on the two legs. In other words, the right-triangle formula is merely a special case of the cosine law.

EXAMPLE 5–6. Solve the triangle when given $a = 12$, $b = 8$, angle $C = 40°$.

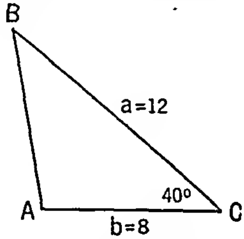


Fig. 5–17.

Solution:

$$c^2 = a^2 + b^2 - 2ab\cos C,$$
$$c^2 = (12)^2 + (8)^2 - (2)(12)(8)(\cos 40°);$$
$$c = \sqrt{144 + 64 - (192)(0.766)},$$
$$c = \sqrt{208 - 147.08} = \sqrt{60.92} = 7.8.$$

The remainder of the solution may be carried through by means of the sine law.

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$
$$\sin A = \frac{a \sin C}{c} = \frac{(12)(\sin 40°)}{7.8},$$
$$\sin A = \frac{(12)(0.6428)}{7.8} = 0.9883,$$
$$A = 98°\,47',$$
$$B = 180 - (A + C) = 41°\,13'.$$

5. Extending the law of cosines. The law of cosines can be extended for use in the solution of problems in the electrical field. Since voltages and currents are treated as vector quantities, it is essential to know how to determine the values for vector sums or vector differences, and this can be done by means of the cosine law.

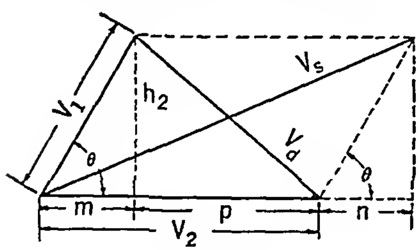


Fig. 5–18.

Consider the two voltages V_1 and V_2 operating at an acute angle θ with each other, as shown in Fig. 5–18, and determine their vector sum and vector difference. Let V_s = vector sum and V_d = vector difference.

It should be noted that the diagonal cutting through the given angle represents the vector sum (V_s) and the diagonal that is opposite the given angle represents the vector difference (V_d).

In Fig. 5–18:

$$V_s^2 = h_1^2 + (V_2 + n)^2,$$
$$V_s^2 = h_1^2 + V_2^2 + 2V_2 n + n^2.$$

Now $\qquad h_1^2 + n^2 = V_1^2$

and $\qquad n = V_1 \cos\theta.$

Then, by substitution:

$$V_s^2 = V_1^2 + V_2^2 + 2V_1V_2 \cos \theta.$$

Again, in Fig 5–18:

$$V_d^2 = h_2^2 + p^2.$$

But $\quad p = V_2 - m.$

Thus $\quad V_d^2 = h_2^2 + V_2^2 - 2V_2m + m^2.$

Now $\quad h_2^2 + m^2 = V_1^2$

and $\quad m = V_1 \cos \theta.$

Then, by substitution,

$$V_d^2 = V_1^2 + V_2^2 - 2V_1V_2 \cos \theta.$$

An inspection of these two formulas will show that the only difference between the expression for the vector sum of two quantities and the expression for the vector difference is in the sign of the last term. For a vector sum, the sign of this term is written plus and for a vector difference the sign is written minus. It should be remembered, however, that the value of the cosine is negative between 90° and 270°, and therefore an obtuse angle for θ will reverse the signs when the value of $\cos \theta$ is inserted.

EXAMPLE 5–7. Given the two voltages of 10 and 12, respectively, operating at an angle of 30°, determine their vector sum and vector difference.

Solution:

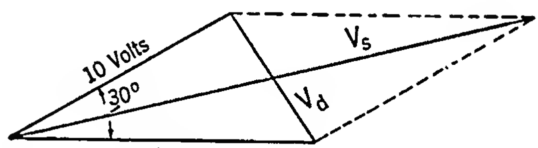


Fig. 5–19.

$$V_s^2 = 10^2 + 12^2 + (2)(10)(12)(\cos 30°),$$
$$V_s^2 = 100 + 144 + (240)(0.866),$$
$$V_s^2 = 100 + 144 + 207.84 = 451.84,$$
$$V_s = \sqrt{451.84} = 21.26.$$
$$V_d^2 = 10^2 + 12^2 - (2)(10)(12)(\cos 30°),$$
$$V_d^2 = 100 + 144 - (240)(0.866),$$
$$V_d^2 = 36.16,$$
$$V_d = \sqrt{36.16} = 6.01.$$

EXERCISE 5–3

Solve for all unknown parts in each of the following:

1. $a = 5.0000,\quad b = 3.0000,\quad C = 45°.$
2. $a = 5,\quad c = 7,\quad B = 30°.$
3. $a = 3,\quad b = 6,\quad c = 5.$
4. $a = 15,\quad b = 112,\quad c = 113.$
5. $b = 8,\quad c = 12,\quad A = 44°.$
6. $a = 0.72,\quad b = 0.31,\quad c = 0.54.$

7. It is desired to find the distance between two points B and C. A third point D is chosen accessible from either B or C. The distances DB and DC are measured and found to be 1,100 ft and 1,750 ft, respectively. The angle at the point D is found to be $16° 50'$. Find the distance BC.

8. An observer is 150 ft from one end of a pond and 175 ft from the other end. The angle subtended by the pond is found to be 120°. Find the length of the pond.

9. Develop the cosine law expression for side a for the case in which angle A is acute but angle C is obtuse. Sides b and c and angle A are given.

10. Given the two voltages of 110 and 75 acting at an angle of $85° 15'$ with each other, determine the vector sum and the vector difference.

11. Given two forces F_1 and F_2 making an obtuse angle A with each other, develop an expression for their vector sum.

12. Develop an expression for the vector difference of the two forces of Problem 11.

13. In Problem 11, find the value of the vector sum if $F_1 = 9.8$, $F_2 = 7.3$, and $A = 115° 25'$.

14. In Problem 12, find the value of the vector difference if $F_1 = 12.5$, $F_2 = 11.4$, and $A = 125° 10'$.

6. Area of a triangle. We found in Chapter 2 that the area of any triangle is equal to one half the product of the base and altitude. If the area is denoted by K, the base by b, and the altitude by h, we have

$$K = \frac{1}{2} bh.$$

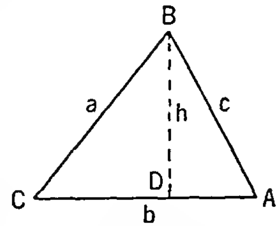


Fig. 5-20.

However, it is not necessary to determine the altitude, since this expression can be converted into one in which an angle is taken into account. For example, the two sides a and b and the angle C included between them are given in the triangle ABC, Fig. 5-20.

Now the area $K = \frac{1}{2} bh$.

But $h = a \sin C$, from the right triangle BCD.

Therefore, by substitution,

$$K = \frac{1}{2} ab \sin C.$$

In like manner

$$K = \frac{1}{2} ac \sin B,$$

$$K = \frac{1}{2} bc \sin A.$$

If the given angle of the triangle is obtuse as in Fig. 5–21, the same result is obtained for the area of the triangle.

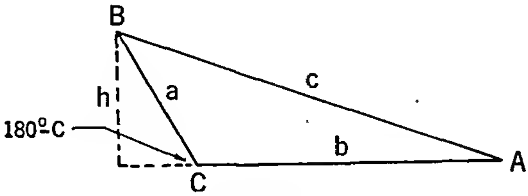


Fig. 5–21.

$$K = \frac{1}{2} bh.$$

But $$h = a \sin (180° - C) = a \sin C.$$

Therefore $$K = \frac{1}{2} ab \sin C.$$

From these expressions, others can be developed by the use of the sine law.

Thus by the sine law,

$$\frac{b}{a} = \frac{\sin B}{\sin A},$$

or $$b = \frac{a \sin B}{\sin A}.$$

By substitution in the expression $\frac{1}{2} ab \sin C$,

$$K = \left(\frac{1}{2}\right)(a)\left(\frac{a \sin B}{\sin A}\right) \sin C,$$

or $$K = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

In a similar way

$$K = \frac{b^2 \sin A \sin C}{2 \sin B},$$

$$K = \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Therefore, we may say that the area of a triangle is determined in one of the three following ways:

1. The area of a triangle is equal to one half the product of the base and altitude.

2. The area of a triangle is equal to one half the product of any two sides multiplied by the sine of the included angle.

3. The area of a triangle is equal to one half the square of any side multiplied by the product of the sines of the adjacent angles and divided by the sine of the opposite angle.

EXAMPLE 5–8. Solve for the area of the triangle ABC when given $a = 9$, $b = 10$, $C = 45°$.

Solution:

$$K = \frac{1}{2}ab \sin C,$$

$$K = \left(\frac{1}{2}\right)(9)(10)(\sin 45°),$$

$$K = (45)(0.707) = 31.815.$$

EXAMPLE 5–9. Solve for the area of the triangle ABC when given $b = 12$, $A = 64°$, $C = 50°$.

Solution:

$$B = 180° - (64° + 50°) = 66°.$$

$$K = \frac{b^2 \sin A \sin C}{2 \sin B},$$

$$K = \frac{(12)^2(\sin 64°)(\sin 50°)}{2 \sin 66°},$$

$$K = \frac{(144)(0.8988)(0.766)}{(2)(0.9135)} = 54.3.$$

EXERCISE 5–4

Solve the following triangles for all unknown parts including the area:

1. $A = 47° 36'$, $B = 65° 42'$, $c = 28.1$.
2. $b = 275.6$, $c = 198.7$, $A = 57° 30'$.
3. $A = 63° 40'$, $C = 69° 4'$, $c = 12.94$.
4. $a = 5.67$, $b = 6.45$, $C = 98° 12'$.
5. $B = 17° 18'$, $C = 61° 5'$, $c = 13.75$.
6. $a = 11.12$, $c = 9.88$, $B = 11° 45'$.

EXERCISE 5–5

Solve the following triangles for all unknown parts including the area:

1. $a = 17.4$, $b = 13.2$, $A = 48° 16'$.
2. $a = 72.6$, $b = 117.5$, $A = 80°$.
3. $a = 17.7$, $b = 21.65$, $A = 35° 36'$.
4. $a = 74.8$, $b = 37.5$, $B = 63° 36'$.

5. Two triangles are possible from the following values: $a = 21.46$. $b = 16.01$, $B = 31° 52'$. Find the difference in the areas without solving for the area of either triangle.

6. Two points, A and B, are not visible from each other but each is visible from a third point C. The distances AC and BC and the angle ACB were measured and found to be 1,930 ft, 1,150 ft, and $47° 45'$, respectively. Find the distance AB.

7. A tower is situated on a hill which inclines $17° 42'.6$ to the horizontal. The angle of elevation of the top of the tower from a point on the

hill is found to be 52° 16′.4. At another point 100 ft farther down the hill and in the same vertical plane with the first point and the tower, the angle of elevation is 42° 27′.3. Find the height of the tower.

8. Two towers on a horizontal plane are 120 ft apart. Halfway between the towers the angle of elevation of the top of one is found to be the complement of the angle of elevation of the top of the other. A person standing successively at the bases of the towers finds the angle of elevation of one is two times that of the other. Find the height of the towers.

9. Two voltages, 125 and 115, respectively, make an angle of 135° with each other. Find their vector sum and also their vector difference.

10. A current of 15 amp combines with an unknown current at an angle of 30° to give a resultant of 25 amp. Find the unknown current.

11. Two sides of a parallelogram are 12.09 in. and 15.16 in. long and the angle between is 56° 58′. Find the length of each of the diagonals. Also, find the area of the parallelogram.

12. One side of a triangle is equal to 27 in. and the adjacent angles are each 30°. Find the radius of the circumscribed circle.

13. The bank of a stream has a slope of 1 in 3½. A surveyer goes 60 yds up the bank and then finds the angle of depression of an object on the opposite bank at the water's edge is 2° 20′. What is the width of the stream?

14. From a certain point on a horizontal plane the angle of elevation of the top of a tower is 62° 57′; at a point 500 ft farther away and in the same vertical plane the angle of elevation is 32° 16′. What is the height of the tower?

15. The angle of elevation of the top of a tower from a point due south is 52° 25′; 100 yds east of this point the angle of elevation is 50° 17′. What is the height of the tower?

16. A body is acted upon by a force of 952 lb and by a force of unknown size making an angle of 46° 28′ with the direction of the first force. The resultant is found to be 1,235 lb. Find the unknown force and the angle it makes with the resultant.

17. Develop an expression for the height of an object standing on an inclined plane, by measuring the angles of elevation M and N, respectively, from two points a certain distance d apart on the surface of the plane, and by measuring the angle A of the inclined plane.

18. The sides of a triangle are 15, 21, and 26. Find the length of the line bisecting the longest side and drawn from the opposite angle.

19. A boat is 12 mi N 61° W from a certain port. A second boat leaves the same port in the direction S 23° W at the rate of 15 mph. At what rate and in what direction will the first boat have to travel in order to intercept the second boat 9 mi from the port?

20. The perimeter of a triangle is 196. The angle at C is half that at B and the angle at B is half that at A. Find the sides of the triangle.

21. A voltage of 110 combines with an unknown voltage at an angle of 35° to give a resultant of 220. Find the value of the unknown voltage and the angle it makes with the resultant.

22. In a certain parallelogram, one side a, one diagonal x, and the angle A between the two diagonals are given. Find the other diagonal, y.

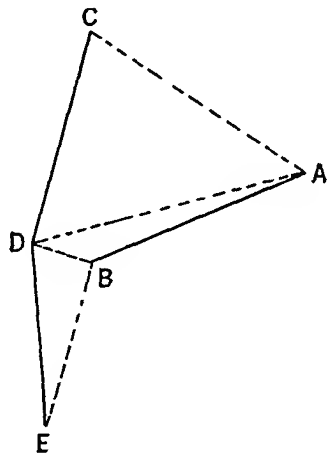
23. In Problem 22, if $a = 3.5$, $x = 6.3$, and $A = 21° 35'$, find y.

24. An unknown voltage V combined vectorially with a known voltage of V_1 at an angle of $\theta°$ gives a resultant of $V_R = 160$. Find the value of the unknown and the angle it makes with the resultant.

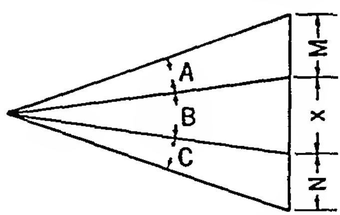
25. In Problem 24, if $V_1 = 100$, $V_R = 160$, and $\theta = 33° 17'$, determine the value of V and the angle it makes with V_R.

26. In the figure below it is desired to find the distance between the two inaccessible points A and B and the following values are measured:

$CD = 944.4$ ft, $ADB = 32° 15'$,
$DE = 673.3$ ft, $BDE = 67° 35'$,
$ACD = 72° 10'$, $DEB = 19° 16'$.
$CDA = 60° 18'$,

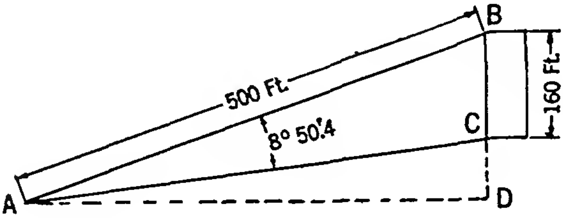


27. Develop an expression for the length of the line x in the following figure, given the lines M and N and the angles A, B, and C.

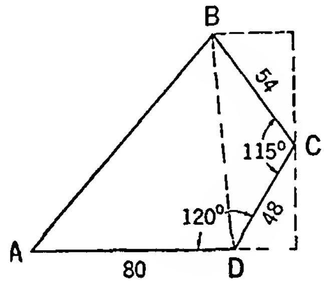


28. In Problem 27, determine the measurement of x if $m = 125$ ft, $n = 92$ ft, $A = 28° 16'$, $B = 34° 25'$, $C = 22° 10'$.

29. Find AD, AC, and angle DAC.



30. Find area $ABCD$.



REVIEW EXERCISE 5-6

Solve the following triangles for all unknown parts including the area. Where two sides and the angle opposite one of them are given, check for the number of solutions.

1.	$b = 43.67$;	$A = 64° 44'.9$;	$C = 37° 25'.6$.
2.	$a = 2.006$;	$A = 40° 32'.3$;	$B = 42° 27'.5$.
3.	$c = 3754.8$;	$A = 94° 27'.8$;	$C = 32° 3'.4$.
4.	$a = 254.96$;	$B = 25° 37'.6$;	$C = 110° 19'.4$.
5.	$b = 197.35$;	$A = 82° 30'.8$;	$B = 35° 37'.3$.
6.	$c = 23.181$;	$A = 101° 35'.5$;	$B = 63° 45'.6$.
7.	$a = 406.27$;	$b = 283.69$;	$B = 39° 18'.7$.
8.	$a = 65.278$;	$b = 56.268$;	$A = 78° 16' 18''$.
9.	$a = 102.31$;	$b = 153.25$;	$A = 16° 23' 42''$.
10.	$b = 1230$;	$c = 1568$;	$B = 56° 12' 18''$.
11.	$a = 175.02$;	$c = 215.54$;	$C = 36° 25' 22''$.
12.	$a = 13.254$;	$b = 5.763$;	$B = 31° 3'.2$.
13.	$a = 34.69$;	$c = 53.97$;	$B = 95° 17'.7$.
14.	$b = 46.327$;	$c = 59.468$;	$A = 78° 13' 45''$.
15.	$a = 76.314$;	$b = 421.05$;	$C = 46° 42' 15''$.
16.	$b = 40.103$;	$c = 47.185$;	$A = 67° 32'.6$.
17.	$a = 18.64$;	$c = 32.38$;	$B = 64° 24'.4$.
18.	$a = 24.63$;	$b = 15.48$;	$C = 13° 50'.1$.
19.	$b = 26.39$;	$c = 58.43$;	$A = 92° 14'.8$.
20.	$a = 23.185$;	$c = 34.279$;	$B = 122° 35' 50''$.

21. $a = 3.724$;	$b = 5.689$;	$C = 104° 12'.4$.
22. $a = 527.30$;	$c = 458.85$;	$C = 57° 46' 54''$.
23. $b = 7.2861$;	$c = 4.3962$;	$C = 27° 50'.4$.
24. $a = 13.658$;	$b = 21.296$;	$C = 58° 10' 48''$.
25. $b = 382.79$;	$c = 528.82$;	$A = 116° 15'.5$.
26. $a = 14.64$;	$b = 21.40$;	$c = 34.73$.
27. $a = 23.81$;	$b = 26.17$;	$c = 18.94$.
28. $a = 47.472$;	$b = 98.104$;	$c = 92.397$.
29. $a = 0.53861$;	$b = 0.47253$;	$c = 0.68352$.
30. $a = 0.002154$;	$b = 0.003247$;	$c = 0.004321$.

31. From Figures 5–1 and 5–2, derive three relationships of the form: $c = a \cos B + b \cos A$.

32. A hill has a perpendicular rise of 72.5 ft for each 100 ft of length of its slope. The angle of elevation of the top from a point 300 ft from the base is 31° 17′.8. What is the height of the hill?

33. A triangular lot is located on the corner of two streets whose angle of intersection is not a right angle. The frontage on one street is 290 ft and on the other is 255 ft. The back line of the lot is 352 ft long. If an adjoining lot is purchased that adds 270 ft to the side with a frontage of 290 ft, how much is the lot increased in area?

34. From a point A, 50 ft from the bank of a stream, the angles of depression of the near and far banks are 14° 57′.2 and 7° 23′.8 respectively. Determine the width of the stream and the elevation of point A above the stream.

35. To determine the shore line frontage (AB) of a quadrangular lot ($ABCD$), the sides BC, CD and DA are measured and found to be 235.4 ft, 156.5 ft, and 104.3 ft, respectively. The angles DAC and DBC are measured and found to be 33° 22′.3 and 28° 45′.6. Find the length of the shore line.

36. Two adjacent sides of a parallelogram are 81.52 and 52.08 and the angle between them is 58° 43′.5. Find the length of each of the diagonals.

37. The bases of a trapezoid are 47.35 ft and 95.74 ft. The angles at the ends of the longer base are 62° 53′.5 and 71° 45′.8. What are the lengths of the other sides?

38. The perpendicular from the vertex C to the base AB of a triangle is equal to 28.5 in. The perimeter of the triangle is 98 in. and the angle at A is 48° 50′. Solve the triangle for all unknown parts.

39. A voltage of 83.42(V_1) combines with another voltage of 110.5 (V_2) at an angle of 42° 37′.6. Determine the vector sum (V_s) and the vector difference (V_d). Place V_1 along the horizontal axis.

40. A voltage of 76.32(V_1) combines with an unknown voltage (V_2) to give a resultant voltage (V_s) of 153.7. The angle between V_1 and V_s is 22° 13′.4. Consider V_1 to be placed along the horizontal axis. Find

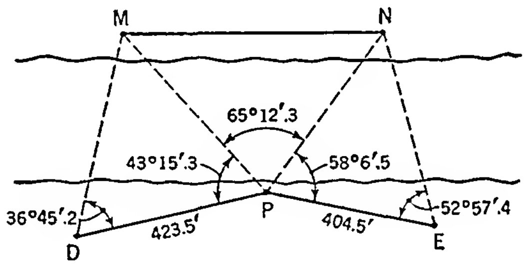
V_2 and the angle it makes with V_1. Also determine the vector difference (V_d) of V_1 and V_2.

41. What will be the vector sum of two currents of 10.62 amp and 12.43 amp, respectively, with an angle of 37° 16′.8 between them?

42. A current of 43.17 amps at an angle of 26° 21′.2 to the horizontal is composed of a current of 27.12 amps at an angle of 53° 18′.7 to the horizontal and an unknown current. Find the unknown and the angle it makes with the other current.

43. The area of a triangle is given as 127.6 sq ft. If angle C = 67° 50′.5 and side a = 87.984 ft, what are the other values for the triangle?

44. Two points, M and N, on the same side of a stream, are inaccessible. They can be seen from a point P somewhere between M and N but on the opposite side. From a point D, on the same side as P and 423.5 ft from P, M and P are sighted. The angles MDP and MPD are measured and found to be 36° 45′.2 and 43° 15′.3 respectively. From another point E, 404.5 feet from D, N and P are sighted. The angles NEP and EPN are measured and found to be 52° 57′.4 and 58° 6′.5. The angle MPN is also measured as 65° 12′.3. Determine the distance from M to N.



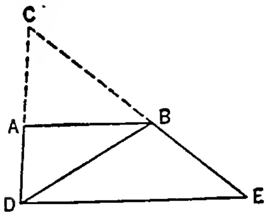
45. A tower is located at the top of a hill. At a point 150.25 ft down the hill from the base of the tower, the angle subtended by the tower is 34° 43′.8. At another point 125.4 ft farther down the hill, the subtended angle is 25° 12′.7 Determine the height of the tower and the angle of inclination of the hill.

46. Two forces give a resultant of 45.75 lb. One of the forces is 21.37 lb and makes an angle of 25° 49′ with the resultant. Determine the other force and the angle between the two forces.

47. From a point A on the south bank of a river that flows due east and is 1.75 miles broad, a connection by bridge and road is to be made to a town B, 3 miles back in a straight line from the north bank. B lies N 40° 25′ E from A. A bridge can be built from point A to either of the two points C and D located on the north bank and a road can be built from either C or D to B. C is N 22° 15′ W from A whereas D is N 41° 35′ E from A. If the bridge costs \$5,000 per mile and the road \$1500 per mile, which route will be cheaper to build and by how much?

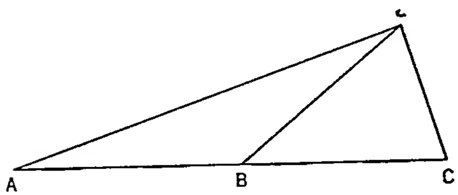
48. Two diagonals of a parallelogram are 57.85 in. and 64.29 in.

and the angle between them is 44° 45′.6. Find the sides of the parallelogram and its area.



49. Find AC and BC in the figure wherein

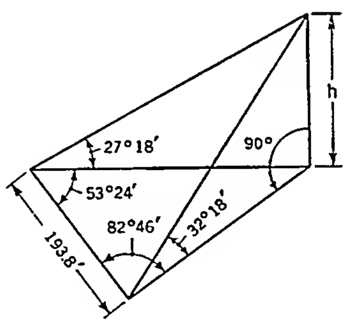
$$AD = 150.2 \text{ ft},$$
$$BE = 249.8 \text{ ft},$$
$$AB = 278.9 \text{ ft},$$
$$BD = 315.6 \text{ ft},$$
$$DE = 497.8 \text{ ft}.$$



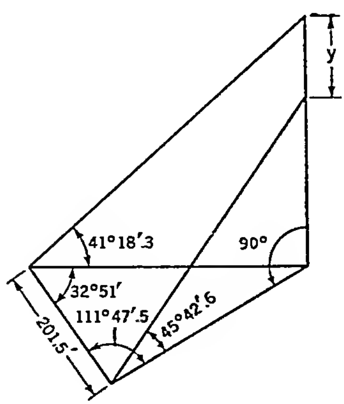
50. Find the distance LA in the figure wherein

$$AB = 237.6 \text{ ft},$$
$$BC = 219.5 \text{ ft},$$
$$ABL = 142° \ 35',$$
$$ACL = 76° \ 18'.$$

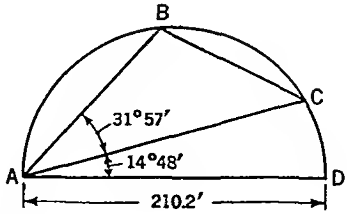
51. Find the distance h.



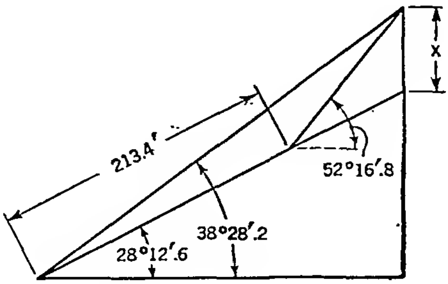
52. Find y.



53. Find the chord BC.



54. Find the distance x.



Chapter 6

ADDITIONAL TRIGONOMETRIC PRINCIPLES AND APPLICATIONS

THE material studied thus far has given us essential methods for the solution of all types of triangles. Any trigonometric problem can be solved by a proper application of the methods considered in Chapters 1 to 5, inclusive. However, there are times when these methods, and especially the cosine law, are cumbersome and difficult to use. For such occasions, other methods that simplify the work have been developed. These methods do not supplant the sine law and cosine law but rather supplement them.

1. The law of tangents. Since the law of cosines is not adapted for the use of logarithms, some other means must be employed if a logarithmic solution is desired. For this condition, the law of tangents has been developed. The law of tangents states:

The difference between two sides of a triangle is in the same ratio to their sum as the tangent of half the difference between the opposite angles is to the tangent of half their sum.

The proof of this law follows. Since

$$\frac{a}{b} = \frac{\sin A}{\sin B}$$

by the law of sines, then by the theory of proportion,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

But in Chapter 4 we learned that

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$$

Therefore,

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$$

Now by merely changing the letters, we obtain the other two forms of this law of tangents:

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)}$$

and

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

In the solution of a triangle where two sides, such as a and b, and the included angle C are known, there is a choice of two methods of solving. The side c can be determined by the law of cosines and the angles A and B by the law of sines. Or the sum of the angles A and B can be found by subtracting the given angle C from 180°. Then, since $a + b$ and $a - b$ are known from the given data, the value for the difference of the angles, $A - B$, can be found by the law of tangents. From the values for $A + B$ and $A - B$, the angles A and B can be found algebraically. This latter method is usually the simpler one, especially where the given sides are not easily squared. When b is larger than a the formula should be written

$$\frac{b-a}{b+a} = \frac{\tan\frac{1}{2}(B-A)}{\tan\frac{1}{2}(B+A)}.$$

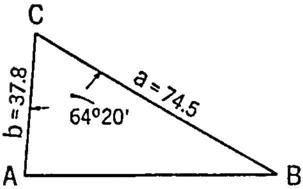


Fig. 6-1.

This will avoid negative numbers in the results.

EXAMPLE 6-1. Solve the triangle ABC by the law of tangents when given $a = 74.5$, $b = 37.8$, and $C = 64° 20'$.

Solution:

$$a + b = 74.5 + 37.8 = 112.3.$$
$$a - b = 74.5 - 37.8 = 36.7.$$
$$A + B = 180° - C = 180° - 64° 20' = 115° 40',$$
$$\frac{A+B}{2} = \frac{115° 40'}{2} = \frac{114° 100'}{2} = 57° 50'.$$

Now

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)},$$

or

$$\tan\frac{1}{2}(A-B) = \frac{(a-b)\tan\frac{1}{2}(A+B)}{a+b},$$

$$\tan\frac{1}{2}(A-B) = \frac{36.7 \tan 57° 50'}{112.3}.$$

$$
\begin{aligned}
\log 36.7 &= 11.5647 - 10 \\
\log \tan 57^\circ 50' &= \underline{0.2014} \\
& 11.7661 - 10 \\
\log 112.3 &= \underline{2.0504} \\
\log \tan \frac{1}{2}(A - B) &= 9.7157 - 10. \\
\frac{1}{2}(A - B) &= 27^\circ 30', \\
\frac{A + B}{2} &= 57^\circ 50', \\
\frac{A - B}{2} &= 27^\circ 30'.
\end{aligned}
$$

Solving these last two algebraically

$$
\begin{aligned}
A + B &= 115^\circ 40' \\
A - B &= \underline{55^\circ\ 0'} \\
2A &= 170^\circ 40' \\
A &= 85^\circ 20'.
\end{aligned}
$$

Therefore,

$$B = 85^\circ 20' - 55^\circ = 30^\circ 20'.$$

Side c may be determined now by use of the sine law. Thus,

$$\frac{a}{c} = \frac{\sin A}{\sin C},$$

or

$$
\begin{aligned}
c &= \frac{a \sin C}{\sin A}, \\
c &= \frac{74.5 \sin 64^\circ 20'}{\sin 85^\circ 20'}. \\
\log 74.5 &= 1.8722 \\
\log \sin 64^\circ 20' &= \underline{9.9549 - 10} \\
& 11.8271 - 10 \\
\log \sin 85^\circ 20' &= \underline{9.9986 - 10} \\
\log c &= 1.8285, \\
c &= 67.37.
\end{aligned}
$$

The solution then is: $A = 85^\circ 20'$; $B = 30^\circ 20'$; and $c = 67.37$.

2. The cotangent law. A further equation that may be used in the solution of an oblique triangle, when two sides and the included angle are given, is known as the *cotangent law.*

Consider the triangles in Fig. 6–2a and Fig. 6–2b, in which sides a and b and angle C are given values.

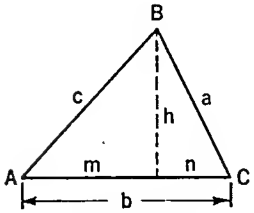


Fig. 6–2a.

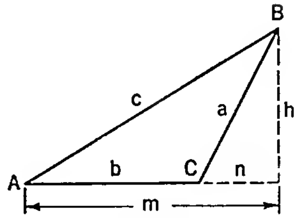


Fig. 6–2b.

$$\frac{h}{a} = \sin C$$

or $$h = a \sin C$$

and $$\frac{n}{a} = \cos C$$

or $$n = a \cos C$$

$$m = b - n = b - a \cos C.$$

$$\frac{h}{a} = \sin (180° - C)$$

$$h = a \sin (180° - C)$$

$$h = a \sin C$$

$$\frac{n}{a} = \cos (180° - C)$$

$$n = a \cos (180° - C)$$

$$n = -a \cos C$$

$$m = b + n = b - a \cos C.$$

Now in either triangle:

$$\cot A = \frac{m}{h} = \frac{b - a \cos C}{a \sin C}$$

or $$\cot A = \frac{b}{a \sin C} - \frac{a \cos C}{a \sin C} = \frac{b}{a} \csc C - \cot C.$$

By erecting the altitude, h, from side a to the opposite vertex, A, it can be shown in the same manner that:

$$\cot B = \frac{a}{b} \csc C - \cot C.$$

Close inspection of these equations will show that the angle required is opposite the side used in the denominator of the first term. With this in mind the equation can be written to fit any problem wherein two sides and the included angle are given. Thus, if sides b and c and angle A are given, the two equations would be:

$$\cot B = \frac{c}{b} \csc A - \cot A,$$

$$\cot C = \frac{b}{c} \csc A - \cot A.$$

EXAMPLE 6–2. Solve Example 6–1 by use of the cotangent law.

Solution:

$$\cot A = \frac{b}{a}\csc C - \cot C,$$

$$= \frac{37.8}{74.5 \sin 64^\circ 20'} - \cot 64^\circ 20',$$

$$= \frac{37.8}{(74.5)(0.9013)} - 0.4806$$

$$= 0.5628 - 0.4806 = 0.0822 \text{ (by slide rule)}$$

$$A = 85^\circ 20'$$

$$\cot B = \frac{a}{b}\csc C - \cot C$$

$$= \frac{74.5}{(37.8)(0.9013)} - 0.4806 = 2.19 - 0.4806 = 1.7094$$

$$B = 30^\circ 20'$$

A further application of the cotangent can be used to determine the altitude of the triangle.

Thus, in the same Figures 6–2a and 6–2b:

	Fig. 6–2a	Fig. 6–2b
	$b = m + n,$	$b = m - n,$
but	$m = h \cot A,$	$m = h \cot A,$
and	$n = h \cot C.$	$n = h \cot (180^\circ - C) = -h \cot C.$
Therefore	$b = h \cot A + h \cot C,$	$b = h \cot A - (-h \cot C),$
and	$h = \dfrac{b}{\cot A + \cot C}.$	$b = h \cot A + h \cot C.$
		$h = \dfrac{b}{\cot A + \cot C}$

The other forms of this altitude equation would be:

$$h = \frac{a}{\cot B + \cot C}$$

and

$$h = \frac{C}{\cot A + \cot B}.$$

EXERCISE 6–1

Solve the following triangles for all unknown parts by use of the law of tangents; check the angles by the cotangent law.

1. $a = 79.4$; $b = 84.5$; $C = 73^\circ 16'$.
2. $b = 874$; $c = 625$; $A = 78^\circ 33'$.
3. $a = 2.45$; $c = 1.732$; $B = 36^\circ 17'$.
4. $a = 13.72$; $b = 11.35$; $C = 15^\circ 47'.3$.
5. $c = 305.1$; $b = 164.7$; $A = 82^\circ 4'.6$.
6. $c = 47.95$; $a = 32.96$; $B = 165^\circ 17'$.

7. Two sides of a triangle are 10 and 12 and the included angle is 40°. Find the third side.

8. Two forces of 835 lb and 572 lb each act upon a body making an angle of 68° 34′ with each other. Find the magnitude and direction of the resultant force.

9. Two voltages of 110 each act at an angle of 120° with each other. Determine the value and direction of the resultant voltage.

10. It is desired to find the distance between two objects, A and B, separated by a marsh. A station C is chosen from which both A and B are accessible. The distances AC, BC, and the angle BCA are measured. They are found to be respectively 425.7 ft, 384.8 ft, and 57° 51′. Find the distance AB.

11. Two cars start from a crossroad point where the two roads make an angle of 45° with each other. One car travels at the rate of 50 mph and the other at the rate of 45 mph. How far apart are they at the end of half an hour?

12. To what form will the equation

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

reduce under each of the following conditions:

(1) the triangle is isosceles with $a = b$;
(2) $C = 90°$;
(3) the triangle is equiangular.

3. The half-angle formulas. If the three sides of a triangle are given, it is possible to find the angles by use of the cosine law. For example, since

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

then

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

However, this expression is not adapted to work with logarithms and for this reason its form has been changed so that logarithms may be used.

By subtracting each side of this equation from unity there results

$$\begin{aligned} 1 - \cos C &= 1 - \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{2ab - a^2 - b^2 + c^2}{2ab} \\ &= \frac{c^2 - (a^2 - 2ab + b^2)}{2ab} \\ &= \frac{c^2 - (a-b)^2}{2ab} \\ &= \frac{(c - a + b)(c + a - b)}{2ab}. \end{aligned}$$

Also by adding each side of the equation to unity, there results

$$1 + \cos C = 1 + \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{2ab + a^2 + b^2 - c^2}{2ab}$$

$$= \frac{(a + b)^2 - c^2}{2ab}$$

$$= \frac{(a + b - c)(a + b + c)}{2ab}.$$

Therefore we have the two equations

$$1 - \cos C = \frac{(b + c - a)(a + c - b)}{2ab} \tag{1}$$

and

$$1 + \cos C = \frac{(a + b - c)(a + b + c)}{2ab}. \tag{2}$$

Now since the perimeter P of a triangle is equal to the sum of the sides, we have

$$a + b + c = P.$$

Then $b + c - a = a + b + c - 2a = P - 2a.$

Likewise $a + c - b = a + b + c - 2b = P - 2b$

and $a + b - c = a + b + c - 2c = P - 2c.$

By substitution, then in Equations (1) and (2),

$$1 - \cos C = \frac{(P - 2a)(P - 2b)}{2ab}, \tag{3}$$

$$1 + \cos C = \frac{(P - 2c)(P)}{2ab}. \tag{4}$$

From a previous chapter we know that

$$\sin \frac{1}{2} C = \sqrt{\frac{1 - \cos C}{2}}$$

and

$$\cos \frac{1}{2} C = \sqrt{\frac{1 + \cos C}{2}}.$$

Therefore,

$$\sin^2 \frac{1}{2} C = \frac{1 - \cos C}{2},$$

or

$$1 - \cos C = 2 \sin^2 \frac{1}{2} C.$$

Also,

$$\cos^2 \frac{1}{2} C = \frac{1 + \cos C}{2},$$

or

$$1 + \cos C = 2 \cos^2 \frac{1}{2} C.$$

Then by substituting these values for $1 - \cos C$ and $1 + \cos C$ in Equations (3) and (4),

$$2 \sin^2 \frac{1}{2} C = \frac{(P - 2a)(P - 2b)}{2ab},$$

or $$\sin^2 \frac{1}{2} C = \frac{(P - 2a)(P - 2b)}{4ab}. \tag{5}$$

and $$2 \cos^2 \frac{1}{2} C = \frac{(P - 2c)(P)}{2ab},$$

or $$\cos^2 \frac{1}{2} C = \frac{(P - 2c)(P)}{4ab}. \tag{6}$$

Equations (5) and (6) can be used for finding the angles of a triangle in which the three sides are given but they may be further simplified by using the semiperimeter s of the triangle in place of the perimeter. Thus, since $P = 2s$, Equation (5) may be written

$$\sin^2 \frac{1}{2} C = \frac{(2s - 2a)(2s - 2b)}{4ab}$$
$$= \frac{2(s - a)(2)(s - b)}{4ab}$$
$$= \frac{4(s - a)(s - b)}{4ab}.$$
$$= \frac{(s - a)(s - b)}{ab}.$$

Therefore, $$\sin \frac{1}{2} C = \sqrt{\frac{(s - a)(s - b)}{ab}}. \tag{7}$$

Also, Equation (6) may be written

$$\cos^2 \frac{1}{2} C = \frac{(2s - 2c)(2s)}{4ab}$$
$$= \frac{2(s - c)(2)(s)}{4ab}$$
$$= \frac{(s - c)(s)}{ab}.$$

Therefore, $$\cos \frac{1}{2} C = \sqrt{\frac{(s)(s - c)}{ab}}. \tag{8}$$

Equations (7) and (8) are called half-angle formulas and are in the form generally found in most texts.

It should be noted that in Equation (7) the sides a and b are used in solving for $\sin \frac{1}{2} C$ whereas in Equation (8) the sides a, b, and c are used in solving for $\cos \frac{1}{2} C$.

Since $$\tan x = \frac{\sin x}{\cos x},$$

it is evident from Equations (7) and (8) that

$$\tan \frac{1}{2} C = \frac{\sin \frac{1}{2} C}{\cos \frac{1}{2} C}$$

$$\tan\frac{1}{2}C = \frac{\sqrt{\dfrac{(s-a)(s-b)}{ab}}}{\sqrt{\dfrac{(s)(s-c)}{ab}}}$$

$$= \sqrt{\frac{(s-a)(s-b)}{(s)(s-c)}}. \tag{9}$$

By merely changing the letters in Equations (7), (8), and (9) the following are obtained:

$$\sin\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad \sin\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}},$$

$$\cos\frac{1}{2}A = \sqrt{\frac{(s)(s-a)}{bc}}, \qquad \cos\frac{1}{2}B = \sqrt{\frac{(s)(s-b)}{ac}},$$

$$\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{(s)(s-a)}}, \qquad \tan\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{(s)(s-b)}}.$$

The sign before the radical is always positive since one half of any angle of a triangle is an acute angle.

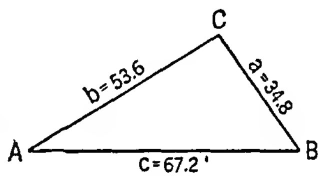


Fig. 6–3.

EXAMPLE 6–3. Solve the triangle ABC, when given the three sides $a = 34.8$; $b = 53.6$; and $c = 67.2$.

Solution:

$$s = \frac{P}{2} = \frac{34.8 + 53.6 + 67.2}{2} = \frac{155.6}{2} = 77.8,$$

$$s - a = 77.8 - 34.8 = 43.0,$$

$$s - b = 77.8 - 53.6 = 24.2,$$

$$s - c = 77.8 - 67.2 = 10.6,$$

$$\sin\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{(43)(24.2)}{(34.8)(53.6)}}.$$

$$\begin{aligned}
\log 43 &= 1.6335 \\
\log 24.2 &= 1.3838 \\
\operatorname{colog} 34.8 &= 8.4584 - 10 \\
\operatorname{colog} 53.6 &= 8.2708 - 10 \\
&\ 2)\overline{19.7465 - 20} \\
\log\sin\frac{1}{2}C &= 9.8733 - 10, \\
\frac{1}{2}C &= 48^\circ\, 19'\, 30'', \\
C &= 96^\circ\, 39'.
\end{aligned}$$

$$\sin\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}},$$

$$\sin\frac{1}{2}B = \sqrt{\frac{(43)(10.6)}{(34.8)(67.2)}}.$$

$$
\begin{aligned}
\log 43 &= 1.6335 \\
\log 10.6 &= 1.0253 \\
\operatorname{colog} 34.8 &= 8.4584 - 10 \\
\operatorname{colog} 67.2 &= \underline{8.1726 - 10} \\
&\ 2)\overline{19.2898 - 20} \\
\log \sin \frac{1}{2} B &= 9.6449 - 10,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2} B &= 26^\circ\, 12', \\
B &= 52^\circ\, 24'. \\
A &= 180^\circ - (B + C), \\
A &= 180^\circ - (96^\circ\, 39' + 52^\circ\, 24'), \\
A &= 30^\circ\, 57'.
\end{aligned}
$$

4. A check on the angles. The expressions for $\tan \frac{1}{2} A$, $\tan \frac{1}{2} B$, and $\tan \frac{1}{2} C$ may be put in more convenient form and used for checking purposes. By multiplying both numerator and denominator of Equation (9) by $s - c$, there results

$$
\begin{aligned}
\tan \frac{1}{2} C &= \sqrt{\frac{(s-a)(s-b)(s-c)}{(s)(s-c)(s-c)}} \\
&= \frac{1}{s-c} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.
\end{aligned}
$$

Now by putting
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$
the equation becomes

$$\tan \frac{1}{2} C = \frac{r}{s-c}.$$

By changing the letters we get

$$\tan \frac{1}{2} A = \frac{r}{s-a}$$

and
$$\tan \frac{1}{2} B = \frac{r}{s-b}.$$

This form of the equation for the tangent is very convenient to use, since the value for r must be determined but once in order to solve for the three angles.

EXAMPLE 6-4. Check the solution for the problem of Example 6-2 by use of the tangent.

Solution:

$$s = \frac{34.8 + 53.6 + 67.2}{2} = \frac{155.6}{2} = 77.8,$$

$$
\begin{aligned}
s - a &= 77.8 - 34.8 = 43, \\
s - b &= 77.8 - 53.6 = 24.2,
\end{aligned}
$$

$$s - c = 77.8 - 67.2 = 10.6,$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{(43)(24.2)(10.6)}{77.8}}.$$

$$\begin{aligned}
\log 43 &= 1.6335 \\
\log 24.2 &= 1.3838 \\
\log 10.6 &= 1.0253 \\
\text{colog } 77.8 &= \underline{8.1090 - 10} \\
\log r^2 &= 12.1516 - 10 \\
&\ \ 2)\underline{2.1516} \\
\log r &= 1.0758, \\
r &= 11.907.
\end{aligned}$$

Now $$\tan \frac{1}{2} A = \frac{r}{s-a} = \frac{11.907}{43.0},$$

$$\begin{aligned}
\log 11.907 &= 1.0758 \\
\text{colog } 43 &= \underline{8.3665 - 10} \\
\log \tan \frac{1}{2} A &= 9.4423 - 10, \\
\frac{1}{2} A &= 15^\circ\ 28'\ 30'', \\
A &= 30^\circ\ 57'.
\end{aligned}$$

$$\tan \frac{1}{2} B = \frac{r}{s-b} = \frac{11.907}{24.2},$$

$$\begin{aligned}
\log 11.907 &= 1.0758 \\
\text{colog } 24.2 &= \underline{8.6162 - 10} \\
\log \tan \frac{1}{2} B &= 9.6920 - 10, \\
\frac{1}{2} B &= 26^\circ\ 12', \\
B &= 52^\circ\ 24'.
\end{aligned}$$

$$\tan \frac{1}{2} C = \frac{r}{s-c} = \frac{11.907}{10.6},$$

$$\begin{aligned}
\log 11.907 &= 1.0758 \\
\text{colog } 10.6 &= \underline{8.9747 - 10} \\
\log \tan \frac{1}{2} C &= 10.0505 - 10, \\
\frac{1}{2} C &= 48^\circ\ 19'\ 30'', \\
C &= 96^\circ\ 39'.
\end{aligned}$$

EXERCISE 6–2

Solve the following triangles by use of the half-angle formulas for sin and cos and check each by the tan:

1. $a = 17.25$; $b = 28.36$; $c = 34.72$.
2. $a = 9.73$; $b = 4.87$; $c = 10.88$.
3. $a = \sqrt{5}$; $b = \sqrt{6}$; $c = 2\sqrt{2}$.

4. $a = 112.6$; $b = 144.8$; $c = 43$.
5. $a = 29$; $b = 78$; $c = 100$.
6. $a = 75$; $b = 62$; $c = 115$.
7. A body is acted upon by two forces of 625 lb and 515 lb each. The resultant force is 718.9 lb. Find the angle between the first two forces and the angles between each of these forces and the resultant.
8. A voltage of 45 combines with another of 75 to give a resultant of 120. Find the angle between the first two and the angles the resultant makes with each of the others.

5. Area of a triangle. In Chapter 5 we developed expressions for finding the area of a triangle when two sides and the included angle are given. These expressions are as follows:

$$K = \frac{1}{2} ab \sin C,$$

$$K = \frac{1}{2} ac \sin B,$$

$$K = \frac{1}{2} bc \sin A.$$

Also, we developed expressions for finding the area of a triangle when given one side and any two angles. These are as follows:

$$K = \frac{a^2 \sin B \sin C}{2 \sin A},$$

$$K = \frac{b^2 \sin A \sin C}{2 \sin B},$$

$$K = \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Now when three sides are given, the above expressions cannot be used directly, since at least one of the angles must be determined first.

However, by use of the half-angle formulas, we can develop certain expressions that will solve for the area of a triangle directly in terms of the three sides.

Since $\sin 2C = 2 \sin C \cos C$

we may write it thus,

$$\sin C = 2 \sin \frac{1}{2} C \cos \frac{1}{2} C.$$

Now the area of a triangle is

$$K = \frac{1}{2} ab \sin C.$$

Substituting the value for $\sin C$,

$$K = \left(\frac{1}{2} ab\right)\left(2 \sin \frac{1}{2} C \cos \frac{1}{2} C\right),$$

or

$$K = ab \sin \frac{1}{2} C \cos \frac{1}{2} C.$$

But from our half-angle formulas

$$\sin \tfrac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

and
$$\cos \tfrac{1}{2} C = \sqrt{\frac{(s)(s-c)}{ab}}.$$

Substituting these values in the expression for the area,

$$K = ab\left(\sqrt{\frac{(s-a)(s-b)}{ab}}\right)\left(\sqrt{\frac{(s)(s-c)}{ab}}\right).$$

Therefore,
$$K = ab\sqrt{\frac{(s)(s-a)(s-b)(s-c)}{(ab)^2}},$$

$$K = \sqrt{(s)(s-a)(s-b)(s-c)}.$$

By multiplying and dividing by s, there results

$$K = \sqrt{\frac{(s^2)(s-a)(s-b)(s-c)}{s}}$$

$$= s\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

or
$$K = rs, \text{ where } r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

EXAMPLE 6–5. Determine the area of the triangle of Example 6–3, in which the sides are given as $a = 34.8$, $b = 53.6$, and $c = 67.2$.

Solution:

$$s = \frac{34.8 + 53.6 + 67.2}{2} = \frac{155.6}{2} = 77.8,$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{(43)(24.2)(10.6)}{77.8}} = 11.907.$$

Therefore,
$$K = rs,$$
$$K = (11.907)(77.8).$$
$$\log 11.907 = 1.0758$$
$$\log 77.8 = 1.8910$$
$$\log K = 2.9668,$$
$$K = 926.4.$$

EXERCISE 6–3

1 to 6. Solve Problems 1 to 6, inclusive, of Exercise 6–2 for the areas of the triangles.

7. Show that the radius r of a circle inscribed in the triangle ABC is equal to
$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

8. Develop an expression for the radius R of a circle circumscribed about the triangle ABC, in terms of the sides and area.

6. Sector and segment areas of a circle. In Fig. 6–4 let r be the radius of the circle and θ the number of radians in the central angle. Then

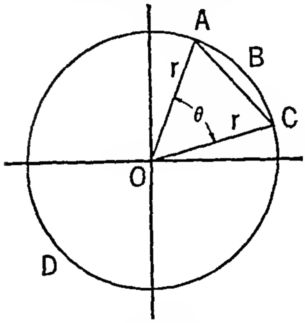


Fig. 6–4.

$$\frac{\text{area of sector } OABC}{\text{area of circle}} = \frac{\theta \text{ radians}}{2\pi \text{ radians}};$$

that is,
$$\frac{K(\text{sector } OABC)}{\pi r^2} = \frac{\theta}{2\pi},$$

from which
$$K(\text{sector } OABC) = \frac{\pi r^2\theta}{2\pi} = \frac{1}{2}r^2\theta.$$

Therefore, the area of a sector of a circle is

$$K(\text{sector}) = \frac{1}{2}r^2\theta,$$

where r is the radius of the circle and θ is the number of radians in the central angle.

The area of the segment ACB is found by subtracting the area of the triangle AOC from the area of the sector $OABC$. Thus,

$$K(\text{segment } ACB) = K(\text{sector } OABC) - K(\text{triangle } AOC),$$

or
$$K(ACB) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta.$$

Therefore $K(\text{segment } ACB) = \frac{1}{2}r^2(\theta - \sin\theta).$

Area of larger segment $ADC = \frac{1}{2}r^2(\Phi - \sin\Phi)$, where $\Phi = 360° - \theta$.

Example 6–6. Find the areas of the sector and segment enclosing a central angle of 35° in a circle whose diameter is 10 in.

Solution.

$$35° = (35)\frac{\pi}{180} \text{ radians}.$$

$$\begin{aligned}
K(\text{sector } OABC) &= \left(\frac{1}{2}\right)(r^2)(\theta) \\
&= \left(\frac{1}{2}\right)(5)^2(35)\frac{\pi}{180} \\
&= \left(\frac{1}{2}\right)(25)(0.6108) \\
K(\text{sector } OABC) &= 7.635 \text{ sq in.}; \\
K(\text{segment } ABC) &= \frac{1}{2}r^2(\theta - \sin\theta) \\
&= \left(\frac{1}{2}\right)(5)^2(0.6108 - \sin 35°) \\
&= \left(\frac{1}{2}\right)(25)(0.6108 - 0.5736) \\
&= (12.5)(0.0372). \\
K(\text{segment } ABC) &= 0.465 \text{ sq in.}
\end{aligned}$$

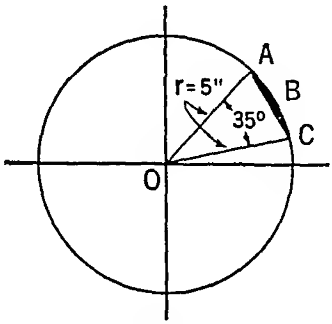


Fig. 6–5.

EXERCISE 6–4

Find the areas of the following sectors and segments of a circle, given:

1. $r = 12$ in.; $\theta = 59° 18'$.
2. $r = 1.2$ ft; $\theta = 138° 16'$.
3. $\theta = 39° 48'$; intercepted arc $= 2$ ft 4 in.
4. $\theta = \frac{\pi}{8}$; intercepted arc $= 9.7$ in.

5. The area of a sector of a circle is 110.45 sq in. and its angle is $65° 19'$. Find the length of the radius and the intercepted arc.

6. Find the area of the larger segment of a circle bounded by a chord 15 in. long at a distance of 6 in. from the center of the circle.

7. A cylindrical oil drum, 5 ft long and 3 ft 6 in. in diameter lies with its axis in a horizontal plane and is filled with oil to a depth of 2 ft. How many cubic feet of oil are in the drum?

8. A cylindrical tank that lies with its axis in a horizontal plane is partly filled with water so that the greatest depth is 14 in. The tank is 18 ft long and 6 ft in diameter. How many gallons of water are in the tank? (231 cu in. = 1 gal.)

7. Equations and identities. In Chapter 3 we studied the simpler forms of identities and equations and have acquired some practice in their solutions. Also, in Chapter 4 we discussed a special type of identity with methods for its solution. But the more complicated identities and equations have not been considered.

In the solutions for the more complicated forms, the same general statements will apply as were found to apply to Chapter 3. The identity should be attacked by taking the more complex side and changing it into terms of simple identities already known, until this more complex side is reduced to the same expression as contained in the less complex side. The solution of the equation generally will be best made by changing all functions into terms of one of the functions only and solving the resulting equation algebraically.

EXAMPLE 6–7. Prove the following identity:

$$\frac{1 + \tan x}{1 - \tan x} = \sec 2x + \tan 2x.$$

Solution: Let us take the left-hand side and work with that only. Thus,

$$\frac{1 + \tan x}{1 - \tan x} = \frac{1 + \dfrac{\sin x}{\cos x}}{1 - \dfrac{\sin x}{\cos x}} = \frac{\cos x + \sin x}{\cos x - \sin x}.$$

This, at first glance, would appear to be as far as we can go. But by resorting to the expedient of multiplying both numerator and denominator

by the denominator with the sign between the terms changed, we are able to simplify still further. This method of attack is used considerably, especially where multiple angles are involved. Therefore,

$$\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)\left(\frac{\cos x + \sin x}{\cos x + \sin x}\right),$$

$$\frac{\cos^2 x + 2 \sin x \cos x + \sin^2 x}{\cos^2 x - \sin^2 x}.$$

Since $\cos^2 x + \sin^2 x = 1,$

$2 \sin x \cos x = \sin 2x,$

and $\cos^2 x - \sin^2 x = \cos 2x.$

By substituting these values there results

$$\frac{1 + \sin 2x}{\cos 2x} = \frac{1}{\cos 2x} + \frac{\sin 2x}{\cos 2x}$$
$$= \sec 2x + \tan 2x.$$

Example 6-8. Solve the following equation for all values of the unknown angle between 0° and 360°:

$$\sin 2x = \cos x.$$

Solution:

Since $\sin 2x = 2 \sin x \cos x,$

then $2 \sin x \cos x = \cos x,$

or $2 \sin x \cos x - \cos x = 0.$

Factoring, $(\cos x)(2 \sin x - 1) = 0.$

Then $\cos x = 0$ and $\sin x = \frac{1}{2}.$

Therefore, $x = 30°; 90°; 150°; 270°.$

By substituting these angles in the original equation, it will be found that all the values are true values.

Example 6-9. Solve the following equation for all values of the unknown angle between 0° and 360°:

$$\sin^2 x + \sin 2x = 1.$$

Solution: $\sin^2 x + \sin 2x = 1.$

Since $\sin^2 x = 1 - \cos^2 x$

and $\sin 2x = 2 \sin x \cos x,$

by substitution there results

$$1 - \cos^2 x + 2 \sin x \cos x = 1,$$

or $2 \sin x \cos x - \cos^2 x = 0.$

Now, by factoring,

$$(\cos x)(2 \sin x - \cos x) = 0,$$

or $\cos x = 0,$

also $2 \sin x - \cos x = 0$

$$2 \sin x = \cos x$$

$$\frac{\sin x}{\cos x} = \frac{1}{2},$$
$$\tan x = \frac{1}{2}.$$

Therefore, $x = 90°; 270°; 26° 34'; 206° 34'.$

By substituting these angles in the original equation, it will be found that all are true values and fit in the original equation.

The equation of Example 6–8 might have been solved in the following manner:

$$\sin^2 x + \sin 2x = 1,$$
$$\sin^2 x + 2 \sin x \cos x = 1,$$
$$2 \sin x\sqrt{1 - \sin^2 x} = 1 - \sin^2 x.$$

Squaring, $4 \sin^2 x - 4 \sin^4 x = 1 - 2 \sin^2 x + \sin^4 x;$

then $5 \sin^4 x - 6 \sin^2 x + 1 = 0.$

Factoring, $(5 \sin^2 x - 1)(\sin^2 x - 1) = 0;$

then $5 \sin^2 x - 1 = 0,$ $\sin^2 x - 1 = 0.$

Whence $\sin^2 x = \frac{1}{5},$ $\sin^2 x = 1,$

and $\sin x = \pm \frac{1}{\sqrt{5}} = \pm \frac{1}{5}\sqrt{5},$ $\sin x = \pm 1.$

Therefore, $x = \sin^{-1} \pm \frac{\sqrt{5}}{5} = \pm 0.4472,$ $x = \sin^{-1} \pm 1,$

$$x = 26° 34'; 90°; 153° 26'; 206° 34'; 270°; 353° 26'.$$

An examination of these two solutions for the equation $\sin^2 x + \sin 2x = 1$ will show that the process of squaring resorted to in the latter method has introduced two more angles—that is, $153° 26'$ and $353° 26'$—that when substituted back in the original equation prove to be values that do not apply.

Therefore, to avoid getting additional answers that do not apply, it is best wherever possible to use other methods than squaring. If squaring seems the only possible method available, then all answers should be checked very carefully to see which one may not apply.

Example 6–10. Solve the following equation for all acute values of the unknown angle:

$$\tan^{-1} x = \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{13}.$$

Solution:

Let $A = \tan^{-1} x$ or $\tan A = x,$

$B = \tan^{-1} \frac{1}{4}$ or $\tan B = \frac{1}{4},$

$C = \tan^{-1} \frac{1}{13}$ or $\tan C = \frac{1}{13}.$

Then $A = B + C.$

Therefore, $$\tan A = \frac{\tan B + \tan C}{1 - \tan B \tan C}.$$

Substituting values,

$$x = \frac{\frac{1}{4} + \frac{1}{13}}{1 - \left(\frac{1}{4}\right)\left(\frac{1}{13}\right)},$$

$$x = \frac{\frac{17}{52}}{1 - \frac{1}{52}} = \frac{\frac{17}{52}}{\frac{51}{52}},$$

$$x = \frac{1}{3} = 0.3333.$$

Therefore, $A = \tan^{-1} 0.3333,$

or $A = 18° 26'.$

EXERCISE 6-5

Prove the following identities:

1. $\tan x = \dfrac{2 \tan \frac{1}{2} x}{1 - \tan^2 \frac{1}{2} x}.$
2. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}.$
3. $\sin x = \dfrac{2 \tan \frac{1}{2} x}{1 + \tan^2 \frac{1}{2} x}.$
4. $2 \sin x + \sin 2x = \dfrac{2 \sin^3 x}{1 - \cos x}.$
5. $\cos^4 2x - \sin^4 2x = \cos 4x.$
6. $\tan x + \dfrac{1}{\tan x} = \dfrac{2}{\sin 2x}.$
7. $\cos 3x = 4 \cos^3 x - 3 \cos x.$
8. $\dfrac{1 - \sin 2x}{\cos 2x} = \dfrac{1 - \tan x}{1 + \tan x}.$

Solve the following equations:

9. $\sin 2x = 2 \cos x.$
10. $\sin x + \cos 2x = 4 \sin^2 x.$
11. $\sin x = \cos 2x.$
12. $\cos x + \cos 2x = 0.$
13. $\tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{7} = \tan^{-1} x.$
14. $\cos 2x = 2 \sin x.$
15. $2 \tan^{-1} \frac{2}{3} - \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{\sqrt{3}x}{3}.$
16. $\sin 2x = 3 \sin^2 x - \cos^2 x.$
17. $\sin^{-1} x + 2 \cos^{-1} x = \frac{2\pi}{3}.$
18. $\sin x + \sin 2x = \sin 3x.$
19. $\cos x - \cos 2x = 1.$
20. $x = 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}.$
21. $4 \cos 2x + 6 \sin x = 5.$

EXERCISE 6-6

Solve the following triangles for all unknown values including the area:

1. $a = 16.71;$ $b = 22.4;$ $c = 19.8.$
2. $a = 0.462;$ $b = 0.630;$ $c = 0.537.$
3. $b = 6.25;$ $c = 4.52;$ $A = 33° 8'.$
4. $a = 4.81;$ $c = 3.13;$ $B = 107° 21'.$

5. From a point $7\frac{1}{2}$ mi from one end of a lake and $4\frac{1}{2}$ mi from the other end, the angle subtended by the lake is 68° 44′ 24″. Find the length of the lake.

6. Find the area of the segment of a circle of radius 14 in. bounded by an arc of 6.8 in.

7. A plot of ground is formed by three intersecting streets. The three sides are 267.5 ft, 350.8 ft, and 325.7 ft, respectively. Find its area.

8. Two forces of 91.5 lb and 83.7 lb, respectively, act upon a body to give a resultant force of 145.8 lb. Find the angles the resultant makes with the other forces.

9. A cylindrical tank with its axis horizontal is filled with water to a depth of 3 ft. The tank is 5 ft in diameter and 14 ft long. Find the weight of the water. (1 cu ft of water weighs 62.4 lb)

10. The currents I_1 and I_2 of the two branches of an alternating-current single-phase circuit have values of 10.7 amp and 8.34 amp, respectively. The angle between them is 39° 30′. Find the resultant current and the angle it makes with each of the others.

11. Two single-phase currents of 6.6 amp and 4.53 amp, respectively, add up to give a line current of 8.71 amp. Find the angles between them and the angles the resultant makes with each.

REVIEW EXERCISE 6-7

Prove the following identities:

1. $\dfrac{1}{\tan x} - \tan x = \dfrac{2}{\tan x}$.

2. $\tan x + \cot x = \dfrac{2}{\sin 2x}$.

3. $\cos x = \cos\left(x + \dfrac{\pi}{3}\right) + \sin\left(x + \dfrac{\pi}{6}\right)$.

4. $\dfrac{1}{\cos^2 2x} = \dfrac{4\tan^2 x}{(1 - \tan^2 x)^2} + 1$.

5. $\sin x + \cos x = \sqrt{2}\cos(x - 45°)$.

6. $\dfrac{\cos 3x}{\cos x} + \dfrac{\sin 3x}{\sin x} = 4\cos 2x$.

7. $\dfrac{\cos x - \sin x}{\cos x + \sin x} = \dfrac{1}{\tan 2x + \sec 2x}$.

8. $\tan x = \dfrac{(1 + \tan^2 x)\sin 2x}{2}$.

9. $\dfrac{1 + \cos 2x}{1 - \cos 2x} = \dfrac{1}{\tan^2 x}$.

10. $\sec^2 x = (2 - \sec^2 x)(\sec 2x)$.

Solve the following equations for all values of the unknown angle that satisfy the given equation:

11. $\sin 2x = \sin x.$
12. $\cos 2x + \cos x + 1 = 0.$
13. $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}.$
14. $2 \tan^{-1} x = \cot^{-1} x.$
15. $\tan 2x = \frac{2}{\tan x}.$
16. $\sin 3x + \sin 2x - \sin x = 0.$
17. $\sin 2x = \cos^2 \left(\frac{\pi}{4} - x\right).$
18. $\sin x + \cos 2x = 1.$
19. $\sin x + \sin 3x = \cos x - \cos 3x.$
20. $\frac{1 + \tan x}{1 - \tan x} = \frac{\sec 2x}{2}.$

Solve the following triangles for all unknown parts including the area:

21. $a = 16.32$; $b = 21.85$; $c = 18.67$.
22. $a = 4.6008$; $b = 3.297$; $C = 106° 27'.8$.
23. $a = 0.6493$; $b = 0.6192$; $c = 0.5427$.
24. $b = 13.05$; $c = 25.24$; $A = 108° 23'.6$.
25. $a = 233.48$; $c = 387.92$; $B = 94° 53'.5$.

26. Two sides of a parallelogram are 19.52 and 17.26. The shorter diagonal is 21.93. Find the longer diagonal and the angles.

27. A cylindrical oil tank is 24 ft long and 6 ft in diameter, and lies with its axis in a horizontal plane. It is filled with oil to a depth of 28 in. Find the number of gallons of oil in the tank and the number of gallons required to fill it. What is the weight of the oil in the tank if 1 cu ft weighs 50 lbs?

28. Three streets intersect so as to form a lot whose measurements are 265.5 ft, 345.2 ft, and 310.7 ft, respectively. Find the angles of intersection and the area of the lot.

29. Two sides of a triangle are 14.36 in. and 21.05 in. The difference between the angles opposite these sides is 20° 8'.2. Solve the triangle for the unknown parts.

30. A segment of a circle is bounded by an arc of 70.5 inches. The diameter of the circle is 32.4 inches. Find the area of the segment.

31. The sector of a circle has an area of 118.5 sq in. The central angle is 122° 5'.6. Find the radius of the circle and the length of the arc.

32. A triangle has sides of 71.63, 43.65, and 82.71. Find the areas of the inscribed and the circumscribed circles.

33. Two forces of 54.82 lbs and 64.13 lbs act at an angle of 47° 42'.6 with each other. Find the magnitude and direction of the resultant.

ANSWERS TO EXERCISES

PART I

Chapter 1—The Slide Rule

Exercise 1-8

17. 92.6%
18. (a) 0.323 (b) 0.625
19. 35.3 miles
20. 1.39 mils
21. (a) 2.94% (b) 1,752 rpm
22. (a) 84.5% (b) 18.3 hp
23. (a) 13.8 mph (b) 22.5 mph
24. (a) 4.91 sq in. (b) 2.93 in.

Chapter 2—Arithmetic with Applications

Exercise 2-1

8. $\frac{217}{1,415}$
9. 268
10. $\frac{215}{1,244}$
11. \$0.81
12. 50,625 lb
13. wt = 44.73 lb; cost = \$8.05
14. 545 tons
15. (a) 54 (b) 99
16. (a) 30 lb (b) \$1.50
17. \$104
18. 10 men

Exercise 2-2

6. $3\frac{4}{9}$
7. $\frac{153}{253}$
8. $\frac{5,328}{17,255}$
9. $\frac{329}{2454}$
10. $\frac{220}{117}$
11. $2\frac{34}{55}$
12. 5 pieces; $5\frac{1''}{4}$ left
13. 24 hr
14. 137
15. $\frac{3}{16}$ in.
16. 120
17. $1\frac{3}{4}$ in.; $2\frac{9}{64}$ in.
18. (a) $1\frac{27}{248}$ lb; $8\frac{99}{100}$ in.
19. 192 sec

Exercise 2-3

9. (a) 12 pieces; (b) 0.58 in.
10. 0.133 in.; $\frac{133}{1,000} = \frac{1}{8}$ in. (approx.)
11. 683.6 sq ft
12. (a) 32 cu ft; (b) 0.0362 lb

Exercise 2-4

4. 12,650 w
5. 1,710 rpm
6. 7.062 lb lead; 0.487 lb tin; 85.95 lb zinc
7. 2.078 in.; 0.144%
8. 118.4 v
9. 75%; 21.2%
10. 5.32%
11. 13.9%; \$42.64
12. 53.2 lb
13. \$177.78

Exercise 2-5

1. $2,113.12
2. $1,159.71
3. $1,261.33
4. $2,531.25
5. $316.14
6. $1,147.50

Exercise 2-6

1. 33 days
2. 8.93 ohms
3. $66\frac{2}{3}$ lb
4. 15 men

Exercise 2-7

10. 1.984 ft
11. 3′ 11.6″
12. 17 ft 6 in.
13. $4\frac{15}{16}$ in.
14. 126.5 ft
15. 142 v
16. 237 ft

Exercise 2-8

1. 1,033.8 gm/sq cm
2. 3 min 10 sec
3. 15.55 miles
4. 0.7646 cu m
5. (a) 51.2 mph; (b) 82.3 km/hr
6. 0.1 cm, or .0394 in.
7. 0.3865 sq mile
8. 0.02697 lb
9. 7.85 gm/cu cm.
10. 1.04 lb
11. (a) 2,720 sq in.; 18.9 sq ft
    (b) $25\frac{5}{16}$ sq in.; 0.1758 sq ft
    (c) 1,550 sq in.; 10.76 sq ft
12. (a) 43,560 sq ft; (b) 4,840 sq yd
13. 0.581%
14. 3×10^{10} cm/sec
15. 12.21 kg/sq meter
16. 3.33
17. 8.92

Chapter 3—The Fundamental Operations in Algebra

Exercise 3-1

1. $16R$
2. $10a - b - 2c + 4d$
3. $8EI$
4. $18x - 8y + 22z$
5. $-2a + 18b - 7c + 16d$
6. $9 + 3r + 2\sqrt{R^2 + x^2} - 3z$
7. $48x^{3/4} + 17y^{2/3} + 19z^{3/5}$
8. $-11x^{1/2} + 9y^{1/2} - 9z$
9. $4x^3 - 2y^2 + 2z^4$
10. $a^3b - 11a^2b^2 + 17ab^3$
11. $3a\sqrt{x} + 9y$
12. $3x^6 + 15.554x^5y - 8x^4y^2 + 5.898x^3y^3 - 12x^2y^4 - 7xy^5 + y^6$
13. $-31a^2 - 28\sqrt{x} - 15\sqrt{ax} + 16$
14. $7x^4\sqrt{a} + (3 + \sqrt{b})x^3y + 7x^2y^2 + 20xy^3 + 5y^4$
15. $\frac{11}{2}a^2 - \frac{25}{4}b^2$
16. $5.7x^3 - 5.6x^2 - 9x + 7$
17. $\frac{3}{40}b - \frac{1}{6}a$
18. $4x + 3y - 15c + 3$
19. $a + b - c + d$
20. $4n - 3m$
21. $p - q + 2s$
22. $3x + 4a - 3b$
23. $15IR - 9IX + 6E - 6$
24. $20IR + 10IX + E$
26. $-2x - 1$
27. $11ax + 8by - 4cz$
28. $2c - x - ab - 9$
29. $4a + 4b - 2c$
30. $4a$
31. $11x + 20y - 40z$
32. $-2a - 3.1b$

Exercise 3-2

21. $x^8 + 2x^4 + 1 - m^4$
22. $a^2 - b^2 - 2bc - c^2$
23. $0.2m^6 - 0.46m^5n + 0.32m^4n^2 - 0.08\,m^3n^3 - 2m^2n^4 + 0.2mn^5 + 1.12n^6$
24. $168c^9 - 211\frac{1}{3}c^8d + 34c^7d^2 + 82c^6d^3 - 139\frac{2}{3}c^5d^4 + 72\frac{11}{18}c^4d^5 + \frac{2}{3}c^3d^6 - \frac{1}{6}c^2d^7$
25. $-0.004x^4 + 0.26x^3y - 2.96x^2y^2 - 1.3xy^3 - 0.1y^4$
26. $0.8 - 14.6y - 25.3y^2 + 6.9y^3 - 0.35y^4$
27. $9a^2 + 12ab + 4b^2 - 16c^2$
28. $x^4 - 4x^2 + 4x - 1$
29. $(x - y)^4 + 4(x - y)^3a + 6(x - y)^2a^2 + 4(x - y)(a^3) + a^4$
30. $(x + y)^3 + 3(x + y)^2(a + b) + 3(x + y)(a + b)^2 + (a + b)^3$
31. $x^4 + x^2y^2 + y^4$
32. $8a^3 - 108a^2b + 486ab^2 - 729b^3$
33. $(y^2 + 4y)$ sq in.
34. $\dfrac{x^2 - 4x}{288}$ sq ft
35. $\dfrac{r}{2}(4x - 2y + \pi r)$
36. $\dfrac{a^2}{16}(48 - \pi)$
37. $15x + 2$
38. $\dfrac{\pi}{4}(64 + 16x - 3x^2)$

Exercise 3-3

1. $x - 4$
2. $y^2 - 3y + 6$
3. $a - 3$
4. $b^2 + 2b + 4$
5. $5x^3 + 4x^2 + 3x + 2$
6. $a^7 - a^6 + 2a^5 - 2a^4 + 3a^3 - 3a^2 + 4a - 1$
7. $y^3 + y^2 + y + 1 + \dfrac{2}{y - 1}$
8. $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5 + \dfrac{2b^6}{a + b}$
9. $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$
10. $1 + y + y^2 + y^3 + y^4 + \dfrac{y^5}{1 - y}$
11. $x^2 - 2x - \dfrac{1}{x^2 - x - 1}$
12. $a^5 + 2a^4 + 4a^3 + 8a^2 + 16a + 32$
13. $y^4 - 3y^3 + 9y^2 - 27y + 81$
14. $14x^2 - 11x + \dfrac{8x - 1}{x^2 + x + 1}$
15. $x^6 + 2x^5 + 2x^4 + 4x^3 + 7x^2 + 14x + 18$
16. $y^4 - 2y^3 + 4y^2 - 8y - 3$
17. $x^2 + xy + y^2$
18. $y^2 - 2y - 3$
19. $4x - 9y$
20. $2a^2 + a - 1 + \dfrac{3a + 4}{a^2 - a + 3}$
21. $y^n + 4$
22. $7x^3 - 5x^2 - 6x - 2$
23. $6x^3 - 10x^2 + 8x - 4$
24. $y^5 - 2y^4 + 4y^3 - 8y^2 + 16y + 6$
25. $a^2 + 5a - 4$
26. $2 - 5x + 2x^2$
27. $3b^2 - 8b + 2$
28. $y^4 + 3y^3 + 4y^2 - 3y - 2$
29. $a^2 + (x - 2n)a + 4n^2 - 2nx$
30. $6x^2 - x - 4$
31. $d^4 - 2d^3 + 3d^2 - 2d + 1$
32. $3a^2 - 2ab - b^2$

Exercise 3-5

1. $\dfrac{3by^2}{2ax}$
2. $\dfrac{2mt^2}{3n}$
3. $a + b$
4. $\dfrac{3x(2x - 3y^2)}{2 - 3y}$
5. $\dfrac{1}{1 + 2x}$
6. $y(a - b)$
7. $\dfrac{2 - a}{a}$
8. $\dfrac{v - 7}{v + 5}$
9. $a + b$
10. $\dfrac{x^2 - x + 1}{x}$
11. $\dfrac{x + y - z}{x - y + z}$
12. $3(3a + 1)$
13. $\dfrac{x + 1}{x}$
14. $\dfrac{(x + y)(x - y)(x^2 + 3y^2)}{(x^2 - xy + y^2)(x^2 + xy + y^2)}$
15. $\dfrac{1}{1 - R}$
16. $\dfrac{R + x}{R(R + 2x)}$

Exercise 3-5 — *Cont.*

17. $\dfrac{x(x^2 + x - 1)}{x^3 + 3x^2 - 1}$
18. $\dfrac{9x - y}{4(x - y)}$
19. x
20. $\dfrac{x^2 + y^2}{xy}$
21. $\dfrac{(x + y + z)^2}{2yz}$
22. $\dfrac{3 + y^2}{y(1 - y^2)}$
23. $\dfrac{R_1 - R_2}{R_1 + R_2}$
24. $\dfrac{3x - x^2 - 3}{2x^2 - 3}$
25. -2
26. $-\dfrac{1}{2x + 1}$
27. $-\dfrac{ab}{a + 1}$
28. $b^y - a^x$
29. $\dfrac{2(x^2 + xy - y^2)}{x^2 - y^2}$
30. $\dfrac{4x^2 - 11x + 2}{x(2x - 1)}$
31. 1
32. $\dfrac{2}{3}(x + 2)$
33. $\dfrac{2}{a - b}$
34. $\dfrac{x + y}{a + b}$
35. $\dfrac{4}{3(x + 1)}$
36. x
37. $\dfrac{(a + 3)^2}{(a^2 + 3a + 9)(a + 7)}$
38. $-\dfrac{b + 5a}{2(a + b)}$
39. $\dfrac{x + y}{x^2 - xy + y^2}$
40. $\dfrac{y^2(x + 6y)}{(x - 5y)(x + 3y)}$
41. $\dfrac{x - 3}{x + 4}$
42. $-\dfrac{1}{ax}$
43. $\dfrac{6c}{b(c - b)}$
44. $\dfrac{t + 3}{t + 1}$
45. $\dfrac{6a + 1}{a - 2}$
46. $\dfrac{2xy}{(3x + 2y)(3x - 2y)}$
47. 1
48. $\dfrac{3(3x - a)}{5a(3a - x)}$
49. $\dfrac{x(a + 2x)}{a^2}$
50. $\dfrac{b}{a}$
51. $\dfrac{x - 4}{x - 5}$
52. $\dfrac{1}{x^2 - y^2}$
53. $\dfrac{2(x^3 + 3x^2 - 2x + 3)}{x(2x^2 + x - 4)}$
54. $\dfrac{5a^2 + 27}{a^2 - 2a + 3}$

Exercise 3-6

27. $5(x + y)\sqrt[3]{x + y}$
28. $3\sqrt{2a - 1}$
29. $3(a - b)\sqrt{3(x + y)(a - b)}$
30. $-4\sqrt{3}$
31. $-2b\sqrt{a - b}$
32. $a^3b^4c^7\sqrt{bc}$
33. $x + y$
34. bc^3d^5
35. 33
36. $6x^2$
37. $\dfrac{x + 6 - 6\sqrt{x - 3}}{x - 12}$
38. $\dfrac{a\sqrt{a + b} - \sqrt{y}}{a^3 + a^2b - y}$
39. $\dfrac{6x + 24 + 26\sqrt{x}}{9x - 16}$
40. $\dfrac{12y + 5\sqrt{xy} - 2x}{9y - 4x}$
41. $4(a + b)\sqrt[3]{2(a + b)^2}$
42. $4\sqrt{a(2a - 1)}$
43. $2(a^2 + b^2)\sqrt{2(a - b)}$
44. $\dfrac{3b - 8\sqrt{ab} + 4a}{9b - 4a}$
45. $\dfrac{a^2 - 2a\sqrt{a - 2} + a - 2}{a^2 - a + 2}$
46. $\dfrac{3b - 11\sqrt{ab} - 4a}{9b - a}$

Exercise 3-7

23. $41 + j13$
24. $6 + j33$
25. $23 - j101$
26. $(a^2 - b^2) + j2ab$
27. $R^2 + x^2$
28. 2
29. $0.1 - j0.3$
30. $-0.1 + j1.3$
31. $\dfrac{x^2 - y^2 - j2xy}{x^2 + y^2}$
32. $-j$
33. $-0.293 - j0.983$
34. $3 - j$
35. $\dfrac{4 + j2y}{4 + y^2}$
36. $0.02 - j1.14$

Chapter 4—Equations and Formulas

Exercise 4-1

7. 3
8. 7
9. 2.25
10. 4
11. 5
12. 8
13. 11
14. 14
15. 21
16. 15
17. 10
18. -3
19. $ab + 7$
20. a or -1

Exercise 4-2

32. $wm = \$52$
    $am = \$25$
33. $A = \$500$
    $B = \$2{,}000$
34. 32
35. 6
36. 16, 18, 20
37. 10, 20, 100
38. 4 mph
39. 16; 29
40. 96 yd
41. 30 dimes
    25 quarters
42. 16 qt; 48 qt
43. 12 yr 4 mo
44. 16 ft
45. 48 min
46. 5 days
47. \$10,500
48. \$15,000
49. \$10,000
50. $d = 6$ in.
51. (a) 15 lb-ft
    (b) $HP = \dfrac{2\pi NT}{550}$
52. $-40°$
53. $f = 60$
54. 46.42 lb-ft
55. 242,400 lb/sq in.
56. 2.35 ohms
57. 18 ohms
58. 19.7 ohms
59. 3.64 amp
60. 9 in.; 11 in.; 15 in.
61. .04 in.
62. (a) 5.26 ohms
    (b) 0.455 in.
    (c) 2,650 ft
65. 1,152 cu ft nitrogen
    288 cu ft oxygen
66. 4 yr 8 mo
67. (a) 37.5 ohms
    (b) 95.83 ohms
    (c) 20 ohms
    (d) 200 ohms
69. 49.25° C

Exercise 4-3

1. $x = 7; y = 2$
2. $x = 8; y = 3$
3. $x = 5; y = 6$
4. $a = 5; b = 8$
5. $x = 3; y = 7$
6. $m = 9; n = 1$
7. $x = 2; y = 14$
8. $x = 8; y = 7$
9. $x = 16; y = 8$
10. $a = 20; b = 60$

Exercise 4-3 — *Cont.*

11. $I_1 = 3; I_2 = 4$
12. $a = 1; b = 3$
13. $I_1 = 24.68; I_2 = -16.06$
14. $x = \dfrac{c^2 + 2c + 2d + d^2}{c + d}$
    $y = \dfrac{c^2 + d^2}{c + d}$
15. $x = n + m$
    $y = n - m$
16. $x = 1; y = 1$
17. $a = 1; b = 2; c = 3; d = 4$
18. $a = \frac{1}{3}; b = \frac{1}{2}; c = 1; d = -1$
19. $a = 48; b = 120; c = 240$
20. $x = 12; y = 20; z = 8$
21. $u = 6; x = 2; y = 3; z = 5$
22. $u = 4; x = 5; y = 6; z = 7$
23. $I_1 = 1.765A; I_2 = 1.27A; I_3 = 3.04A$
24. 9; 16
25. 8; 3
26. $\frac{5}{9}$
27. A = \$40; B = \$32
28. M = \$8; A = \$4
29. 250 cu ft; 200 cu ft
30. A = \$12,800; B = \$7,200
31. $m = 375; w = 125$
32. 108 v
34. 255 mph; 195 mph
35. 0.24 ohm
36. F = 5.1 ft; R = 6.36 ft
37. w = 30 yd; l = 40 yd; path = 5 yd
38. silk = \$2.00 per yd
    cloth = \$1.50 per yd
39. 32 miles
40. Base = 32 ft
    Alt. = 28 ft
41. 340.2 yd/sec; 8.7 mph
42. 5%; 8%
43. c = 420 ft; V_1 = 50 ft/sec;
    V_2 = 40 ft/sec
44. 15.4 lb tin; 24.6 lb lead
45. A = 36 days; B = 45 days; C = 36 days;
    $D = 25\frac{5}{7}$ days; together = $8\frac{4}{7}$ days

Exercise 4-4

1. $m = 6$
   $n = -5$
2. $x = 4$
   $y = 6$
3. $x = 2$
   $y = 3$
   $z = 4$
4. $x = 3$
   $y = -4$
   $z = 5$
5. $I_1 = 0.255$
   $I_2 = 2.904$
   $I_3 = 3.159$

Exercise 4-5

1. $x = \pm 4$
2. $x = \pm 5$
3. $x = \pm 3$
4. $x = \pm 1$
5. $x = \pm 12$
6. $x = \pm 11$
7. $x = \pm 3$
8. $x = \pm 12$
9. $x = \pm 2$
10. $x = \pm 5$
11. $x = 3$ or -5
12. $x = 3$ or 9
13. $x = 5$ or -6
14. $x = -6$
15. $x = \frac{6}{7}$ or $-\frac{5}{12}$
16. $x = \pm 5$ or ± 3
17. $x = 4$
18. $x = a$ or $3a$
19. $x = \pm 4$ or ± 7
20. $x = -a$ or $+b$
21. $x = 8$ or -2
22. $x = 16$ or -10
23. $x = 5$ or 21
24. $x = \frac{2}{3}$ or $-\frac{3}{2}$
25. $x = 4$ or $-\frac{2}{5}$
26. $x = \frac{7}{2}$ or -5
27. $x = 2$ or $-\frac{4}{5}$
28. $x = 1$ or -3
29. $x = 2$ or $-\frac{5}{7}$
30. $x = 2b$ or $-2a$
31. $x = 4$ or 6
32. $y = 5$ or -4
33. $A = 11$ or 16
34. $x = 26$ or -19
35. $E = -2 \pm \sqrt{5}$
36. $x = a$ or $-b$
37. 16 yd
38. 16 ohms or 9 ohms
39. 7 mph

Exercise 4-5 — *Cont.*

40. $t = 4$ sec
41. $h = 14.13$ in.
43. 96 min
44. 354 ft
45. 3.575 amp
46. $\frac{1}{2}$
47. 4.3 hr; 2.3 hr
48. 2.929 in.
49. $x = 0.649$ or -1.849
 $y = 3.947$ or -3.547
50. $x = 3.236$ or -1.236
 $y = 1.236$ or -3.236

Exercise 4-6

1. $x = 5$
2. $x = 6$
3. $a = 0$ or $\frac{10}{21}$
4. $E = 0$ or 8
5. $y = 10$
6. $x = \pm \frac{2a\sqrt{3}}{3}$
7. $x = b^2$ or $-\frac{25b^2}{3}$
8. $a = \pm \frac{5}{8}$
9. $x = 144$
10. $x = 2a$
11. $x = 2.25$ or 159.75
12. $x = 1.47b$ or $0.432b$

Exercise 4-7

1. $V = 904.7$ cu in.
2. 9 : 16
3. $l = 4.34$ in.
4. $L = 3555\frac{5}{9}$ lb
5. 36 mph
6. $2\frac{1}{4}$ in.
7. 1 : 4
8. 2.09 ohms

Chapter 5—Graphical Representation

Exercise 5-1

1. 5; 8.25; 7.81; 5.38
2. 8.60; 10; 7.62; 9.49
3. (b) 22.2° C; 158° F
7. $-\frac{1}{2}$; $+2$
8. 1; 3
9. -3; $+\frac{1}{3}$; $+\frac{1}{2}$
10. 2.5

Exercise 5-2

2. $x - 3y + 8 = 0$
3. $x - 2y + 5 = 0$
4. $3x + 2y + 7 = 0$
5. $x + y = 0$

Exercise 5-3

1. $E = \frac{1}{2}R + 0.94$
2. $T = 0.98s - 2{,}941$
3. $P = 0.43315D - 0.0022$
4. $-0.00136t + 0.6365$
5. $1{,}752 - 4.09I$
6. $S = -.0012t + 1.096$

Exercise 5-4

1. $x = 1\frac{2}{7}$; $y = 2\frac{2}{7}$
2. $x = \pm 1$ and ± 2
 $y = \pm 2$ and ± 1
3. $x = 2.06$ and 0.34
 $y = 2.18$ and -2.98
4. $K = \pm 6\sqrt{2}$ for tangent

Chapter 6—Logarithms

Exercise 6-5

1. 10.778
2. 2.595×10^8
3. 224.61
4. 1.2004×10^{-10}
5. 33,556
6. 89.512
7. 6.5438×10^{-3}
8. 1.2835×10^{-4}
9. 289.19
10. 0.048241
11. 0.18348
12. 0.11554
13. 26.828
14. 1.167×10^{-10}
15. 0.54948

Exercise 6-6

1. 1.07375×10^9
2. 4.1035×10^8
3. 8.303×10^{-15}
4. 2.1184
5. 2.8176×10^{11}
6. 1.2117×10^{-13}
7. 5.081×10^{-5}
8. 0.69884
9. 6.762
10. 0.34355
11. 1.4646
12. 1.0234
13. 132.6
14. 0.3833
15. 7.321×10^{-3}
16. 2.07×10^{-3}
17. 0.91514
18. 4.239
19. 15.1025
20. 8.808×10^{-2}

Exercise 6-7

1. 4.8467×10^{-2}
2. 7.3626×10^6
3. 4.5814×10^{-2}
4. 6,535
5. 2.8288×10^{-15}

Exercise 6-8

5. 2.893
6. 0.31546
7. 2.45
8. $x = -0.319$
   $y = 1.581$
10. $t = 0.0208$ sec
12. $t = 0.009148$ sec
13. $\dfrac{x}{y} = 11.774$
14. $x = 3.155$
    $y = -0.713$

Exercise 6-9

1. 203.69 cu in.
2. 2.2163 in.
3. 947.46 grams
4. 154.24 sq in.
5. $r = 25.064$ in.
6. $HP = 157$
7. 5.0958×10^{11}
8. 7.7963×10^{10}
9. $202.25
10. $V = 3{,}363.8$ cu in., or 1.95 cu ft
11. $f = 1.414 \times 10^{-3}$
12. $d = 20.985$ in.
13. $V_2 = 46.417$
14. $V = 35{,}413$
15. $HP = 68.95$
16. $i = 1.175$ amp
17. $Q = 0.1002$ cu ft/sec
18. $R = 5.4787$
19. $d = 0.71143$ in.
20. $I_p + I_g = 0.10353$ amp
21. $I = 0.296$
22. $Pm = 52.257$ lb/sq in.
23. $W = 792.45$
24. $T = 34.253°$
25. $W_h = 46{,}663$ w.
26. $N = 4 \times 10^{-4}$
27. $T_2 = 37.357$ lb

PART II

Chapter 1—Angles and Functions of Angles

Exercise 1-1

32. 7.162 in.
33. 0.0233 in.
34. 4.7124 in.
35. $56\frac{2}{3}\pi$ rad./sec.
36. 1,740 rpm
37. 80
38. $5\frac{1}{2}\pi$
39. $\frac{\pi}{5}$
40. 120π

Exercise 1-2

4. $\sin A = \frac{\sqrt{2}}{2}$; $\cot A = 1$
   $\cos A = \frac{\sqrt{2}}{2}$; $\sec A = \sqrt{2}$
   $\tan A = 1$; $\csc A = \sqrt{2}$
5. $\sin A = \frac{\sqrt{5}}{5}$; $\cot A = 2$
   $\cos A = \frac{2\sqrt{5}}{5}$; $\sec A = \frac{\sqrt{5}}{2}$
   $\tan A = \frac{1}{2}$; $\csc A = \sqrt{5}$
6. $\sin A = 0.60$
   $\cos A = 0.80$
   $\tan A = 0.75$
   $\cot A = 1.333$
   $\sec A = 1.25$
   $\csc A = 1.667$
7. $\sin A = 0.3845$
   $\cos A = 0.923$
   $\tan A = 0.4167$
   $\cot A = 2.4$
   $\sec A = 1.083$
   $\csc A = 2.6$
8. $a = 6$
9. $b = 5.29$
10. $\cos A = .661$
    $\tan A = 1.134$
    $\cot A = 0.882$
    $\sec A = 1.512$
    $\csc A = 1.333$

Exercise 1-3

1. $\sin A = \frac{1}{2} = \cos B$
   $\cos A = \frac{\sqrt{3}}{2} = \sin B$
   $\tan A = \frac{\sqrt{3}}{3} = \cot B$
   $\cot A = \sqrt{3} = \tan B$
   $\sec A = \frac{2\sqrt{3}}{3} = \csc B$
   $\csc A = 2 = \sec B$
3. $A = 45°$
   $A = 18°$
   $A = 22° 30'$
   $A =$ Any value
4. 7.071 in.
5. 16 in.; 30°; 60°
6. 7.5 in.; 12.99 in.

Exercise 1-4

31. $4° 46' 29''$
32. (b) $a = 6.67$
    $b = 7.45$

Chapter 2—The Right Triangle

Exercise 2-1

1. $A = 35°$; $b = 83.26$; $c = 101.64$
2. $B = 46° 32'$; $b = 2.5109$; $c = 3.4596$
3. $B = 13° 44'$; $a = 134.46$; $c = 138.41$
4. $A = 71° 45'$; $a = 51.038$; $c = 53.742$

Exercise 2-1—*Cont.*

5. $A = 21° 26'.4$;	$b = 6.2079$;	$c = 6.6693$
6. $B = 55° 42'.2$;	$b = 111.18$;	$c = 134.58$
7. $A = 63° 45'.6$;	$a = 0.77616$;	$c = 0.86534$
8. $B = 27° 44'.3$;	$a = 0.35313$;	$c = 0.39898$
9. $A = 51° 07' 30''$;	$b = 0.01169$;	$c = 0.018625$
10. $B = 60°$;	$b = 3$;	$c = 3.4641$

Exercise 2-2

1. $B = 53° 08'$;	$a = 2.9998$;	$b = 4.0002$
2. $A = 74° 43'.2$;	$a = 0.01476$;	$b = 0.00403$
3. $A = 17° 41' 20''$;	$a = 28.815$;	$b = 90.35$
4. $B = 65° 14' 15''$;	$a = 0.72815$;	$b = 1.5786$
5. $B = 15°$;	$a = 0.47064$;	$b = 0.12611$
6. $A = 50° 46'$;	$a = 3.7335$;	$b = 3.0486$
7. $A = 28° 41'.4$;	$a = 0.044406$;	$b = 0.081142$
8. $B = 58° 16' 45''$;	$a = 1.0408$;	$b = 1.6838$
9. $B = 39° 10'$;	$a = 6.2591$;	$b = 5.0987$
10. $A = 32° 03'$;	$a = 27.992$;	$b = 44.71$

Exercise 2-3

1. $A = 77° 41'.6$;	$B = 12° 18'.4$;	$a = 7.2562$
2. $A = 53° 7'.8$;	$B = 36° 52'.2$;	$c = 24.15$
3. $A = 57° 30'.8$;	$B = 32° 29'.2$;	$c = 0.89938$
4. $A = 21° 52'.4$;	$B = 68° 07'.6$;	$b = 0.81228$
5. $A = 53° 7'.8$;	$B = 36° 52'.2$	$c = 79$
6. $A = 36° 52'.2$;	$B = 53° 7'.8$;	$b = 0.652$
7. $A = 36° 52'.2$;	$B = 53° 7'.8$	$c = 128.09$
8. $A = 39° 5'.6$;	$B = 50° 54'.4$;	$a = 0.21125$
9. $A = 54° 16'.2$;	$B = 35° 43'.8$;	$a = 69.506$
10. $A = 37° 34'.4$;	$B = 52° 25'.6$;	$b = 1.9317$

Exercise 2-6

1. $F = 120.3$ lb $\theta = 8° 33'$
2. $R = 74.65$ $\theta = 18° 31'$ or $23° 59'$
3. $V_t = 182.3$ v $\theta = 35° 43'$ with 120 v. or $39° 33'$ with 110 v.
4. $V_2 = 64.3$ v $\theta_2 = 30° 9'$
5. 5.176 lb
6. 103.9 lb; 67° 04′

Exercise 2-7

1. 109 ft
2. 141.81 ft
3. bldg. = 239 ft; flagpole = 96.2 ft
4. 51.3 ft
5. N 35° 32′ E 430.1 miles
6. N 48° 05′ E 446.4 miles

Exercise 2-8

1. $A = 38° 59'$;	$C = 99° 33'$;	$b = 0.460$;	Area = 0.09921
2. $B = 40° 37'$;	$C = 114° 07'$;	$a = 4.414$;	Area = 13.56
3. $A = 36° 59'$;	$B = 90° 03'$;	$C = 52° 58'$;	Area = 304.7
4. $A = 41° 23'$;	$B = 85° 46'$;	$C = 52° 51'$;	Area = 2,298
5. $B = 70° 24'$;	$b = 193.1$;	$c = 198.2$;	Area = 10,790
6. $A = 46° 04'$;	$a = 0.1668$;	$b = 0.2079$;	Area = 0.01631

Exercise 2-9

1. 36° 52′
2. 141.4 ft
3. 26.622 ft
4. 26.67 ft
5. 479 ft
6. 45.66 v; 9° 7′.4
7. 9.198 mph
8. 552.5 lb; S 89° 36′.2 E
9. 380.85 lb; S 62° 49′.4 W
10. 32.942 ft; 77.942 ft
12. 573.35 ft
13. 10.218 in.; 31.53 sq in.
14. 218.76 v; 18° 48′.6
15. $a^2 \sin A \cos A$
16. 0.9428
17. 14° 21′.8
18. 24.749 ft; 3,712.3 sq
19. 121.81 ft; 8° 3′.8
20. 1.2855 mi
21. 53.419 ft
22. 57.842 ft
23. 677.08 lb
24. 400 ft
25. 42.72 ft; 69° 26′.7
26. 2,509.2 lb
27. 10 mph; 53° 7′.8
28. $h = m(\cot\theta - \tan\theta)$

Chapter 3—Functions of Angles of Any Magnitude

Exercise 3-6

1. Period $= 2\pi = 360°$
2. Period $= \pi = 180°$
3. Period $= 2\pi = 360°$
4. Period $= 2\pi = 360°$
5. Period $= \pi = 180°$
6. Period $= 6\pi = 1{,}080°$
7. Period = 4 radians = 229° 11′
8. Period $= \dfrac{4\pi}{3} = 240°$

Exercise 3-7

1. 240°; 300°
2. 30°; 330°
3. 144° 44′; 324° 44′
4. 30°; 210°
5. 45°; 135°
6. 105° 01′; 254° 59′
7. 61° 40′; 241° 40′
8. 114° 50′; 294° 50′
9. 136° 34′; 223° 26′
10. 183° 7′; 356° 53′
11. a
12. ±0.70711
13. +.57735
14. 35° 16′; 215° 16′

Exercise 3-9

1. 30°
150°
210°
330°
2. 60°
120°
240°
300°
3. 45°
135°
225°
315°
4. 90°
5. 45°
135°
225°
315°
6. 30°
135°
150°
210°
315°
330°
7. 134° 12′.3
225° 47′.7
8. 74° 13′.8
285° 46′.2
9. 90°
330°
10. 30°, 150°, 45°, 225
11. 30°
150°
270°
12. 90°
120°
240°
270°

Chapter 4—Functions of Two Angles

Exercise 4-1

3. $\sin 90° = 1$
4. $\cos 90° = 0$
5. $\cos 15° = 0.966$
6. $\sin 15° = .259$
7. $\sin 105° = 0.966$
8. $\cos 105° = -0.259$
9. $\sin 135° = 0.707$
10. $\cos 135° = -0.707$
11. $\sin 165° = 0.259$
12. $\cos 165° = -0.966$
14. $\tan 15° = 0.268$
15. $\cot 15° = 3.732$
16. $\tan 90° = \infty$
17. $\cot 90° = 0$
18. $\tan 105° = -3.732$
19. $\cot 105° = -0.268$
20. $\tan 135° = -1$
21. $\cot 135° = -1$
22. $\tan 165° = -0.268$
23. $\cot 165° = -3.732$

Exercise 4-2

4. $\cos(x + y) = -0.508$
   $\tan(x + y) = -1.697$
   $\cot(x + y) = -0.589$
5. $\cos(x - y) = -0.969$
   $\tan(x - y) = 0.254$
   $\cot(x - y) = +3.937$
6. $\sin(A + B) = -0.246$
   $\cos(A + B) = -0.969$
   $\tan(A + B) = +0.254$
   $\cot(A + B) = +3.937$
7. $\sin(A - B) = +0.862$
   $\cos(A - B) = +0.508$
   $\tan(A - B) = +1.697$
   $\cot(A - B) = +0.589$
8. $\sin(A + B) = +0.153$
   $\cos(A + B) = +0.988$
   $\tan(A + B) = +0.1548$
   $\cot(A + B) = +6.462$

Exercise 4-3

4. $\sin 3x = 3 \sin x - 4 \sin^3 x$
5. $\cos 3x = 4 \cos^3 x - 3 \cos x$
6. $\sin 2x = 0.866$
7. $\cos 2x = -0.5000$
8. $\tan 2x = -1.732$
9. $\cot 2x = -0.577$

Exercise 4-5

1. $\sin A = 0.3736$
   $\cos A = 0.9276$
   $\tan A = 0.4026$
   $\cot A = 2.483$

Chapter 5—The Sine and Cosine Laws with Applications

Exercise 5-1

2. $b = 2{,}053.7$
   $c = 2{,}365$
   $C = 123° 15'$
3. $a = 53.809$
   $c = 47.839$
   $A = 108° 55'$
4. $a = 1{,}342.5$
   $b = 1{,}117.9$
   $C = 47° 08'$
5. $b = 585.24$
   $c = 461.79$
   $C = 34° 25'$
6. $a = 623.56$
   $c = 921.3$
   $B = 77° 03'$
7. $b = 2{,}324.9$
   $c = 1{,}629.9$
   $C = 26° 12'$
8. sides = 600; 1,039.3
   altitude = 519.63
9. $\dfrac{(m - n)(\sin B)}{\sin(A + B)}$; $\dfrac{(m - n)\sin A}{\sin(A + B)}$
10. 4.1982; 5.6863
11. $\dfrac{d \sin \Phi}{\sin(\theta + \Phi)}$; $\dfrac{d \sin \theta}{\sin(\theta + \Phi)}$
12. 9.3875; 4.6135
13. 63.489 v
14. $AB = 1{,}325.4$ ft

Exercise 5-2

1. $B = 58° 6'$; $B_1 = 12° 34'$
   $C = 67° 14'$; $C_1 = 112° 46'$
   $b = 8.9498$; $b_1 = 2.2937$
3. $a = 2.79$
   $A = 32° 22'.7$
   $C = 90°$
4. $c = 23.859$
   $B = 48° 19'$
   $C = 64° 9'$
6. $c = 0.80447$
   $A = 31° 21'$
   $C = 107° 8'$
7. $c = 92.024$; $c_1 = 52$
   $A = 46° 22'$; $A_1 = 133° 38'$
   $C = 117° 23'$; $C' = 30° 7'$
8. 8.717 in.
9. $a = 9.7284$ in.
   $b = 9.8970$ in.
   $c = 3.6839$ in.

Exercise 5-3

1. $c = 3.5758$
   $A = 98° 36'.5$
   $B = 36° 23'.5$
2. $b = 3.6577$
   $A = 43° 7'$
   $C = 106° 53'$
3. $A = 29° 56'$
   $B = 93° 49'$
   $C = 56° 15'$
4. $A = 7° 37'.5$
   $B = 82° 22'.5$
   $C = 90°$
5. $a = 8.3599$
   $B = 41° 40'$
   $C = 94° 20'$
6. $A = 112° 59'$
   $B = 23° 21'$
   $C = 43° 40'$
7. 766.48 ft
8. 281.73 ft
10. $V_s = 138.18$; $V_d = 127.9$
13. $F = 9.3766$
14. $F = 21.222$

Exercise 5-4

1. $a = 22.593$
   $b = 27.885$
   $C = 66° 42'$
   $K = 289.31$
2. $a = 237.89$
   $B = 77° 43'.3$
   $C = 44° 46'.7$
   $K = 23{,}097$
3. $a = 12.417$
   $b = 10.176$
   $B = 47° 16'$
   $K = 59.006$
4. $c = 9.175$
   $A = 37° 42'.6$
   $B = 44° 5'.4$
   $K = 18.098$
5. $a = 15.386$
   $b = 4.6712$
   $A = 101° 37'$
   $K = 31.456$
6. $b = 2.48$
   $A = 114° 2'$
   $C = 54° 13'$
   $K = 11.19$

Exercise 5-5

1. $c = 23.13$
   $B = 34° 28'.8$
   $C = 97° 15'.2$
   $K = 113.92$
3. $c = 30.031$; $c_1 = 5.1754$
   $B = 45° 24'$; $B_1 = 134° 36'$
   $C = 99°$; $C_1 = 9° 48'$
   $K = 189.24$; $K_1 = 32.612$
4. $c = 67.133$
   $A = 86° 22'$
   $B = 30° 2'$
   $K = 1256.2$
5. 128.15
6. 1,436.2 ft
7. 146.18 ft
8. $h_1 = 40$ ft; $h_2 = 90$ ft
9. $V_s = 92.308$; $V_d = 221.76$
10. $I = 10.86$ amp
11. 13.273; 23.996; 153.66 sq in.
12. 15.588 in.
13. 346.82 yd
14. 465.88 ft
15. 319.8 yd
16. 368.42; 33° 58'.6
17. $h = \dfrac{d \sin (N - A) \sin (M - A)}{\sin (M - N) \cos A}$
18. 12.8
19. 26.25 mph; S 26° 22' E
20. 87.226; 69.95; 38.82
21. 120.64 v; 16° 40'
22. $x \cos A \pm \sqrt{4a^2 - x^2 \sin^2 A}$

Exercise 5-5—*Cont.*

23. 12.4633
24. $\sqrt{V_R^2 - V_1^2 \sin^2 \theta} - V_1 \cos \theta$
    $\sin^{-1} \dfrac{V_1 \sin \theta}{V_R}$
25. 66.69 v; 20° 3′ 33″
26. 1,037.4 ft
27. $$\frac{-(m+n) \pm \sqrt{(m+n)^2 - 4\,\dfrac{mn \sin A \sin C - mn \sin (A+B) \sin (B+C)}{\sin A \sin C}}}{2}$$
28. 110.6 ft
29. $AC = 353.45$ ft
    $AD = 169.875$ ft
    $DAC = 61° 17'.9$
30. Area = 4,606.6

Chapter 6—Additional Trigonometric Principles and Applications

Exercise 6-1

1. $A = 50° 58' 18''$
   $B = 55° 45' 42''$
   $C = 97.88$
2. $a = 968.3$
   $B = 62° 12' 25''$
   $C = 39° 14' 35''$
3. $b = 1.47$
   $A = 99° 30' 46''$
   $C = 44° 12' 14''$
4. $c = 4.167$
   $A = 116° 23' 32''$
   $B = 47° 49' 10''$
5. $a = 326.1$
   $B = 30° 0' 46''$
   $C = 67° 54' 38''$
6. $b = 80.27$
   $A = 5° 59' 16''$
   $C = 8° 43' 44''$
7. 7.756
8. 1,172; 27° 1′ 16″ from 835 lb
9. 110; 60° from either voltage
10. $AB = 393.65$ ft
11. 18.3 mi

Exercise 6-2

1. $A = 29° 36' 22''$
   $B = 54° 18' 50''$
   $C = 96° 4' 48''$
2. $A = 63° 25' 8''$
   $B = 26° 35' 22''$
   $C = 89° 59' 30''$
3. $A = 49° 29' 42''$
   $B = 56° 23' 56''$
   $C = 74° 6' 22''$
4. $A = 35° 30' 18''$
   $B = 131° 40' 48''$
   $C = 12° 48' 54''$
5. $A = 12° 16' 52''$
   $B = 34° 53' 52''$
   $C = 132° 49' 16''$
6. $A = 36° 37' 41''$
   $B = 29° 33' 5''$
   $C = 113° 49' 14''$
7. $A = 102° 28' 22''$
   $B = 44° 23' 2''$
   $C = 58° 5' 20''$

Exercise 6-3

1. 243.23
2. 23.692
3. 2.634
4. 1,808.1
5. 829.57
6. 2126.9
8. $R = \dfrac{abc}{4K}$

Exercise 6-4

1. sector = 74.52 sq in.
   segment = 12.61 sq in.
2. sector = 1.7375 sq ft
   segment = 1.2582 sq ft
3. sector = 564.33 sq in.
   segment = 44.30 sq in.
4. sector = 119.8 sq in.
   segment = 3.0567 sq in.
5. radius = 13.92 in.
   arc = 15.58 in.
6. 252.16 sq in.
7. 28.42 cu ft
8. 520.62 gal

Exercise 6-5

9. 90°; 270°
10. 30°; 150°; 199° 28′ 17″; 340° 31′ 43″
11. 30°; 150°; 270°
12. 60°; 180°; 300°
13. $x = 0.33363$
14 21° 28′.26; 158° 31′.74
15. $x = 0.879$
16. 45°; 225°
17. $x = 0.866$
18. 0°; 120°; 180°; 240°
19. 60°; 90°; 270°; 300°
20. $x = 45°$; 225°
21. 14° 28′.65; 30°; 150°; 165° 31′.35

Exercise 6-6

1. A = 46° 8′ 42″
   B = 75° 9′ 24″
   C = 58° 41′ 54″
   K = 159.91
2. A = 45° 47′ 12″
   B = 77° 47′ 36″
   C = 56° 25′ 12″
   K = 0.121245
3. B = 101° 48′.6
   C = 45° 3′.4
   a = 3.49
   K = 7.7206
4. b = 6.474
   A = 45° 10′ 3″
   C = 27° 28′ 57″
   K = 7.185
5. 7.2127 mi
6. 1.8522 sq in.
7. 41,370 sq ft
8. 32° 0′ 42″ with 91.5 lb
   35° 24′ 54″ with 83.7 lb
9. 10,762.5 lb
10. 17.942 amp
    17° 12′.1 or 22° 17′.9
11. 78° 38′ 9″; 30° 39′ 25″; 47° 58′ 44″

Tables

CONTENTS

Table I

LOGARITHMS OF NUMBERS

2000–2509

N.	L. 0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

P. P.

2500–3009

N.	L. 0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

	16
1	1.6
2	3.2
3	4.8
4	6.4
5	8.0
6	9.6
7	11.2
8	12.8
9	14.4

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

P. P.

4000–4509

N.	L. 0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

4500–5009

N.	L. 0	1	2	3	4	5	6	7	8	9
450	65 321	331	341	350	360	369	379	389	398	408
451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725	734	744	753	763	772	782	792
455	801	811	820	830	839	849	858	868	877	887
456	896	906	916	925	935	944	954	963	973	982
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66 087	096	106	115	124	134	143	153	162	172
459	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
461	370	380	389	398	408	417	427	436	445	455
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	745	755	764	773	783	792	801	811	820	829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	*006	*015
468	67 025	034	043	052	062	071	080	089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
471	302	311	321	330	339	348	357	367	376	385
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	669	679	688	697	706	715	724	733	742	752
476	761	770	779	788	797	806	815	825	834	843
477	852	861	870	879	888	897	906	916	925	934
478	943	952	961	970	979	988	997	*006	*015	*024
479	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
481	215	224	233	242	251	260	269	278	287	296
482	305	314	323	332	341	350	359	368	377	386
483	395	404	413	422	431	440	449	458	467	476
484	485	494	502	511	520	529	538	547	556	565
485	574	583	592	601	610	619	628	637	646	655
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	806	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
491	108	117	126	135	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	461	469	478	487	496	504	513	522	531	539
496	548	557	566	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

P. P.

5000–5509

N.	L. 0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70 070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	252	260	269	278	286	295	303	312	321
505	329	338	346	355	364	372	381	389	398	406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	*003
513	71 012	020	029	037	046	054	063	071	079	088
514	096	105	113	122	130	139	147	155	164	172
515	181	189	198	206	214	223	231	240	248	257
516	265	273	282	290	299	307	315	324	332	341
517	349	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	*008
525	72 016	024	032	041	049	057	066	074	082	090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	485	493	501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	835	843	852	860	868	876	884	892	900	908
536	916	925	933	941	949	957	965	973	981	989
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
538	73 078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	640	648	656	664	672	679	687	695	703	711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
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5500–6009

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551	115	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	429	437	445	453	461	468	476	484	492	500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	*005	*012	*020	*028	*035	*043
563	75 051	059	066	074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	205	213	220	228	236	243	251	259	266	274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	967	974	982	989	997	*005	*012	*020	*027	*035
576	76 042	050	057	065	072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	185
578	193	200	208	215	223	230	238	245	253	260
579	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	716	723	730	738	745	753	760	768	775	782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	967	975	982	989	997	*004
589	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	452	459	466	474	481	488	495	503	510	517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
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P. P.

6000–6509

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600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
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P. P.

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6500–7009

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650	81 291	298	305	311	318	325	331	338	345	351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
691	948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	055	061	067
693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	198	205	211	217	223	230	236	242	248	255
696	261	267	273	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504
700	510	516	522	528	535	541	547	553	559	566
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P. P.

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P. P.

7000–7509

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700	84 510	516	522	528	535	541	547	553	559	566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	009	016	022	028	034	040	046	052	058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	*004	*010	*016	*022	*028
725	86 034	040	046	052	058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
741	982	988	994	999	*005	*011	*017	*023	*029	*035
742	87 040	046	052	058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	216	221	227	233	239	245	251	256	262	268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558
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P. P.

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P. P.

7500–8009

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751	564	570	576	581	587	593	599	604	610	616	
752	622	628	633	639	645	651	656	662	668	674	
753	679	685	691	697	703	708	714	720	726	731	
754	737	743	749	754	760	766	772	777	783	789	
755	795	800	806	812	818	823	829	835	841	846	
756	852	858	864	869	875	881	887	892	898	904	
757	910	915	921	927	933	938	944	950	955	961	
758	967	973	978	984	990	996	*001	*007	*013	*018	
759	88 024	030	036	041	047	053	058	064	070	076	
760	081	087	093	098	104	110	116	121	127	133	
761	138	144	150	156	161	167	173	178	184	190	6
762	195	201	207	213	218	224	230	235	241	247	1 0.6 2 1.2 3 1.8
763	252	258	264	270	275	281	287	292	298	304	
764	309	315	321	326	332	338	343	349	355	360	4 2.4 5 3.0 6 3.6
765	366	372	377	383	389	395	400	406	412	417	
766	423	429	434	440	446	451	457	463	468	474	
767	480	485	491	497	502	508	513	519	525	530	7 4.2 8 4.8 9 5.4
768	536	542	547	553	559	564	570	576	581	587	
769	593	598	604	610	615	621	627	632	638	643	
770	649	655	660	666	672	677	683	689	694	700	
771	705	711	717	722	728	734	739	745	750	756	
772	762	767	773	779	784	790	795	801	807	812	
773	818	824	829	835	840	846	852	857	863	868	
774	874	880	885	891	897	902	908	913	919	925	
775	930	936	941	947	953	958	964	969	975	981	
776	986	992	997	*003	*009	*014	*020	*025	*031	*037	
777	89 042	048	053	059	064	070	076	081	087	092	
778	098	104	109	115	120	126	131	137	143	148	
779	154	159	165	170	176	182	187	193	198	204	
780	209	215	221	226	232	237	243	248	254	260	
781	265	271	276	282	287	293	298	304	310	315	5
782	321	326	332	337	343	348	354	360	365	371	1 0.5 2 1.0 3 1.5
783	376	382	387	393	398	404	409	415	421	426	
784	432	437	443	448	454	459	465	470	476	481	4 2.0 5 2.5 6 3.0
785	487	492	498	504	509	515	520	526	531	537	
786	542	548	553	559	564	570	575	581	586	592	
787	597	603	609	614	620	625	631	636	642	647	7 3.5 8 4.0 9 4.5
788	653	658	664	669	675	680	686	691	697	702	
789	708	713	719	724	730	735	741	746	752	757	
790	763	768	774	779	785	790	796	801	807	812	
791	818	823	829	834	840	845	851	856	862	867	
792	873	878	883	889	894	900	905	911	916	922	
793	927	933	938	944	949	955	960	966	971	977	
794	982	988	993	998	*004	*009	*015	*020	*026	*031	
795	90 037	042	048	053	059	064	069	075	080	086	
796	091	097	102	108	113	119	124	129	135	140	
797	146	151	157	162	168	173	179	184	189	195	
798	200	206	211	217	222	227	233	238	244	249	
799	255	260	266	271	276	282	287	293	298	304	
800	309	314	320	325	331	336	342	347	352	358	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

8000–8509

N.	L. 0	1	2	3	4	5	6	7	8	9
800	90 309	314	320	325	331	336	342	347	352	358
801	363	369	374	380	385	390	396	401	407	412
802	417	423	428	434	439	445	450	455	461	466
803	472	477	482	488	493	499	504	509	515	520
804	526	531	536	542	547	553	558	563	569	574
805	580	585	590	596	601	607	612	617	623	628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	795	800	806	811	816	822	827	832	838	843
810	849	854	859	865	870	875	881	886	891	897
811	902	907	913	918	924	929	934	940	945	950
812	956	961	966	972	977	982	988	993	998	*004
813	91 009	014	020	025	030	036	041	046	052	057
814	062	068	073	078	084	089	094	100	105	110
815	116	121	126	132	137	142	148	153	158	164
816	169	174	180	185	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	265	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	355	360	365	371	376
820	381	387	392	397	403	408	413	418	424	429
821	434	440	445	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	535
823	540	545	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	635	640
825	645	651	656	661	666	672	677	682	687	693
826	698	703	709	714	719	724	730	735	740	745
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	840	845	850
829	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	944	950	955
831	960	965	971	976	981	986	991	997	*002	*007
832	92 012	018	023	028	033	038	044	049	054	059
833	065	070	075	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	169	174	179	184	189	195	200	205	210	215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	335	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
841	480	485	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	645	650	655	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	845	850	855	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	942	947	952	957	962	967	973	978	983	988
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

P. P.

8500–9009

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
850	92 942	947	952	957	962	967	973	978	983	988	
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
852	93 044	049	054	059	064	069	075	080	085	090	
853	095	100	105	110	115	120	125	131	136	141	
854	146	151	156	161	166	171	176	181	186	192	
855	197	202	207	212	217	222	227	232	237	242	
856	247	252	258	263	268	273	278	283	288	293	
857	298	303	308	313	318	323	328	334	339	344	
858	349	354	359	364	369	374	379	384	389	394	
859	399	404	409	414	420	425	430	435	440	445	
860	450	455	460	465	470	475	480	485	490	495	
861	500	505	510	515	520	526	531	536	541	546	
862	551	556	561	566	571	576	581	586	591	596	
863	601	606	611	616	621	626	631	636	641	646	
864	651	656	661	666	671	676	682	687	692	697	
865	702	707	712	717	722	727	732	737	742	747	
866	752	757	762	767	772	777	782	787	792	797	
867	802	807	812	817	822	827	832	837	842	847	
868	852	857	862	867	872	877	882	887	892	897	
869	902	907	912	917	922	927	932	937	942	947	
870	952	957	962	967	972	977	982	987	992	997	
871	94 002	007	012	017	022	027	032	037	042	047	
872	052	057	062	067	072	077	082	086	091	096	
873	101	106	111	116	121	126	131	136	141	146	
874	151	156	161	166	171	176	181	186	191	196	
875	201	206	211	216	221	226	231	236	240	245	
876	250	255	260	265	270	275	280	285	290	295	
877	300	305	310	315	320	325	330	335	340	345	
878	349	354	359	364	369	374	379	384	389	394	
879	399	404	409	414	419	424	429	433	438	443	
880	448	453	458	463	468	473	478	483	488	493	
881	498	503	507	512	517	522	527	532	537	542	
882	547	552	557	562	567	571	576	581	586	591	
883	596	601	606	611	616	621	626	630	635	640	
884	645	650	655	660	665	670	675	680	685	689	
885	694	699	704	709	714	719	724	729	734	738	
886	743	748	753	758	763	768	773	778	783	787	
887	792	797	802	807	812	817	822	827	832	836	
888	841	846	851	856	861	866	871	876	880	885	
889	890	895	900	905	910	915	919	924	929	934	
890	939	944	949	954	959	963	968	973	978	983	
891	988	993	998	*002	*007	*012	*017	*022	*027	*032	
892	95 036	041	046	051	056	061	066	071	075	080	
893	085	090	095	100	105	109	114	119	124	129	
894	134	139	143	148	153	158	163	168	173	177	
895	182	187	192	197	202	207	211	216	221	226	
896	231	236	240	245	250	255	260	265	270	274	
897	279	284	289	294	299	303	308	313	318	323	
898	328	332	337	342	347	352	357	361	366	371	
899	376	381	386	390	395	400	405	410	415	419	
900	424	429	434	439	444	448	453	458	463	468	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

P. P.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

9000–9509

N.	L. 0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	665	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	785	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895	899
910	904	909	914	918	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96 047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365	369	374
920	379	384	388	393	398	402	407	412	417	421
921	426	431	435	440	445	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	685	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825	830	834	839	844
930	848	853	858	862	867	872	876	881	886	890
931	895	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965	970	974	979	984
933	988	993	997	*002	*007	*011	*016	*021	*025	*030
934	97 035	039	044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	165	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	313	317	322	327	331	336	340	345	350	354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465	470	474	479	483	488	493
944	497	502	506	511	516	520	525	529	534	539
945	543	548	552	557	562	566	571	575	580	585
946	589	594	598	603	607	612	617	621	626	630
947	635	640	644	649	653	658	663	667	672	676
948	681	685	690	695	699	704	708	713	717	722
949	727	731	736	740	745	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	5
1	0.5
2	1.0
3	1.5
4	2.0
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6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

9500–10009

N.	L. 0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855	859
952	864	868	873	877	882	886	891	896	900	905
953	909	914	918	923	928	932	937	941	946	950
954	955	959	964	968	973	978	982	987	991	996
955	98 000	005	009	014	019	023	028	032	037	041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105	109	114	118	123	127	132
958	137	141	146	150	155	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250	254	259	263	268
961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039

N. L. 0 1 2 3 4 5 6 7 8 9

P. P.

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1	0.5
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4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

P. P.

Table II

TRIGONOMETRIC FUNCTIONS TO FOUR PLACES

Degrees	Sine		Tangent		Cotangent		Cosine		
	Value	Log	Value	Log	Value	Log	Value	Log	
0° 00′	.0000	———	.0000	———	———	———	1.0000	.0000	90° 00′
10	.0029	7.4637	.0029	7.4637	343.77	2.5363	1.0000	.0000	50
20	.0058	7.7648	.0058	7.7648	171.89	2.2352	1.0000	.0000	40
30	.0087	7.9408	.0087	7.9409	114.59	2.0591	1.0000	.0000	30
40	.0116	8.0658	.0116	8.0658	85.940	1.9342	.9999	.0000	20
50	.0145	8.1627	.0145	8.1627	68.750	1.8373	.9999	.0000	10
1° 00′	.0175	8.2419	.0175	8.2419	57.290	1.7581	.9998	9.9999	89° 00′
10	.0204	8.3088	.0204	8.3089	49.104	1.6911	.9998	9.9999	50
20	.0233	8.3668	.0233	8.3669	42.964	1.6331	.9997	9.9999	40
30	.0262	8.4179	.0262	8.4181	38.188	1.5819	.9997	9.9999	30
40	.0291	8.4637	.0291	8.4638	34.368	1.5362	.9996	9.9998	20
50	.0320	8.5050	.0320	8.5053	31.242	1.4947	.9995	9.9998	10
2° 00′	.0349	8.5428	.0349	8.5431	28.636	1.4569	.9994	9.9997	88° 00′
10	.0378	8.5776	.0378	8.5779	26.432	1.4221	.9993	9.9997	50
20	.0407	8.6097	.0407	8.6101	24.542	1.3899	.9992	9.9996	40
30	.0436	8.6397	.0437	8.6401	22.904	1.3599	.9990	9.9996	30
40	.0465	8.6677	.0466	8.6682	21.470	1.3318	.9989	9.9995	20
50	.0494	8.6940	.0495	8.6945	20.206	1.3055	.9988	9.9995	10
3° 00′	.0523	8.7188	.0524	8.7194	19.081	1.2806	.9986	9.9994	87° 00′
10	.0552	8.7423	.0553	8.7429	18.075	1.2571	.9985	9.9993	50
20	.0581	8.7645	.0582	8.7652	17.169	1.2348	.9983	9.9993	40
30	.0610	8.7857	.0612	8.7865	16.350	1.2135	.9981	9.9992	30
40	.0640	8.8059	.0641	8.8067	15.605	1.1933	.9980	9.9991	20
50	.0669	8.8251	.0670	8.8261	14.924	1.1739	.9978	9.9990	10
4° 00′	.0698	8.8436	.0699	8.8446	14.301	1.1554	.9976	9.9989	86° 00′
10	.0727	8.8613	.0729	8.8624	13.727	1.1376	.9974	9.9989	50
20	.0756	8.8783	.0758	8.8795	13.197	1.1205	.9971	9.9988	40
30	.0785	8.8946	.0787	8.8960	12.706	1.1040	.9969	9.9987	30
40	.0814	8.9104	.0816	8.9118	12.251	1.0882	.9967	9.9986	20
50	.0843	8.9256	.0846	8.9272	11.826	1.0728	.9964	9.9985	10
5° 00′	.0872	8.9403	.0875	8.9420	11.430	1.0580	.9962	9.9983	85° 00′
10	.0901	8.9545	.0904	8.9563	11.059	1.0437	.9959	9.9982	50
20	.0929	8.9682	.0934	8.9701	10.712	1.0299	.9957	9.9981	40
30	.0958	8.9816	.0963	8.9836	10.385	1.0164	.9954	9.9980	30
40	.0987	8.9945	.0992	8.9966	10.078	1.0034	.9951	9.9979	20
50	.1016	9.0070	.1022	9.0093	9.7882	.9907	.9948	9.9977	10
6° 00′	.1045	9.0192	.1051	9.0216	9.5144	.9784	.9945	9.9976	84° 00′
10	.1074	9.0311	.1080	9.0336	9.2553	.9664	.9942	9.9975	50
20	.1103	9.0426	.1110	9.0453	9.0098	.9547	.9939	9.9973	40
30	.1132	9.0539	.1139	9.0567	8.7769	.9433	.9936	9.9972	30
40	.1161	9.0648	.1169	9.0678	8.5555	.9322	.9932	9.9971	20
50	.1190	9.0755	.1198	9.0786	8.3450	.9214	.9929	9.9969	10
7° 00′	.1219	9.0859	.1228	9.0891	8.1443	.9109	.9925	9.9968	83° 00′
10	.1248	9.0961	.1257	9.0995	7.9530	.9005	.9922	9.9966	50
20	.1276	9.1060	.1287	9.1096	7.7704	.8904	.9918	9.9964	40
30	.1305	9.1157	.1317	9.1194	7.5958	.8806	.9914	9.9963	30
40	.1334	9.1252	.1346	9.1291	7.4287	.8709	.9911	9.9961	20
50	.1363	9.1345	.1376	9.1385	7.2687	.8615	.9907	9.9959	10
8° 00′	.1392	9.1436	.1405	9.1478	7.1154	.8522	.9903	9.9958	82° 00′
10	.1421	9.1525	.1435	9.1569	6.9682	.8431	.9899	9.9956	50
20	.1449	9.1612	.1465	9.1658	6.8269	.8342	.9894	9.9954	40
30	.1478	9.1697	.1495	9.1745	6.6912	.8255	.9890	9.9952	30
40	.1507	9.1781	.1524	9.1831	6.5606	.8169	.9886	9.9950	20
50	.1536	9.1863	.1554	9.1915	6.4348	.8085	.9881	9.9948	10
9° 00′	.1564	9.1943	.1584	9.1997	6.3138	.8003	.9877	9.9946	81° 00′
	Value	Log	Value	Log	Value	Log	Value	Log	
	Cosine		Cotangent		Tangent		Sine		Degrees

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
9° 00′	.1564	9.1943	.1584	9.1997	6.3138	.8003	.9877	9.9946	**81° 00′**
10	.1593	9.2022	.1614	9.2078	6.1970	.7922	.9872	9.9944	50
20	.1622	9.2100	.1644	9.2158	6.0844	.7842	.9868	9.9942	40
30	.1650	9.2176	.1673	9.2236	5.9758	.7764	.9863	9.9940	30
40	.1679	9.2251	.1703	9.2313	5.8708	.7687	.9858	9.9938	20
50	.1708	9.2324	.1733	9.2389	5.7694	.7611	.9853	9.9936	10
10° 00′	.1736	9.2397	.1763	9.2463	5.6713	.7537	.9848	9.9934	**80° 00′**
10	.1765	9.2468	.1793	9.2536	5.5764	.7464	.9843	9.9931	50
20	.1794	9.2538	.1823	9.2609	5.4845	.7391	.9838	9.9929	40
30	.1822	9.2606	.1853	9.2680	5.3955	.7320	.9833	9.9927	30
40	.1851	9.2674	.1883	9.2750	5.3093	.7250	.9827	9.9924	20
50	.1880	9.2740	.1914	9.2819	5.2257	.7181	.9822	9.9922	10
11° 00′	.1908	9.2806	.1944	9.2887	5.1446	.7113	.9816	9.9919	**79° 00′**
10	.1937	9.2870	.1974	9.2953	5.0658	.7047	.9811	9.9917	50
20	.1965	9.2934	.2004	9.3020	4.9894	.6980	.9805	9.9914	40
30	.1994	9.2997	.2035	9.3085	4.9152	.6915	.9799	9.9912	30
40	.2022	9.3058	.2065	9.3149	4.8430	.6851	.9793	9.9909	20
50	.2051	9.3119	.2095	9.3212	4.7729	.6788	.9787	9.9907	10
12° 00′	.2079	9.3179	.2126	9.3275	4.7046	.6725	.9781	9.9904	**78° 00′**
10	.2108	9.3238	.2156	9.3336	4.6382	.6664	.9775	9.9901	50
20	.2136	9.3296	.2186	9.3397	4.5736	.6603	.9769	9.9899	40
30	.2164	9.3353	.2217	9.3458	4.5107	.6542	.9763	9.9896	30
40	.2193	9.3410	.2247	9.3517	4.4494	.6483	.9757	9.9893	20
50	.2221	9.3466	.2278	9.3576	4.3897	.6424	.9750	9.9890	10
13° 00′	.2250	9.3521	.2309	9.3634	4.3315	.6366	.9744	9.9887	**77° 00′**
10	.2278	9.3575	.2339	9.3691	4.2747	.6309	.9737	9.9884	50
20	.2306	9.3629	.2370	9.3748	4.2193	.6252	.9730	9.9881	40
30	.2334	9.3682	.2401	9.3804	4.1653	.6196	.9724	9.9878	30
40	.2363	9.3734	.2432	9.3859	4.1126	.6141	.9717	9.9875	20
50	.2391	9.3786	.2462	9.3914	4.0611	.6086	.9710	9.9872	10
14° 00′	.2419	9.3837	.2493	9.3968	4.0108	.6032	.9703	9.9869	**76° 00′**
10	.2447	9.3887	.2524	9.4021	3.9617	.5979	.9696	9.9866	50
20	.2476	9.3937	.2555	9.4074	3.9136	.5926	.9689	9.9863	40
30	.2504	9.3986	.2586	9.4127	3.8667	.5873	.9681	9.9859	30
40	.2532	9.4035	.2617	9.4178	3.8208	.5822	.9674	9.9856	20
50	.2560	9.4083	.2648	9.4230	3.7760	.5770	.9667	9.9853	10
15° 00′	.2588	9.4130	.2679	9.4281	3.7321	.5719	.9659	9.9849	**75° 00′**
10	.2616	9.4177	.2711	9.4331	3.6891	.5669	.9652	9.9846	50
20	.2644	9.4223	.2742	9.4381	3.6470	.5619	.9644	9.9843	40
30	.2672	9.4269	.2773	9.4430	3.6059	.5570	.9636	9.9839	30
40	.2700	9.4314	.2805	9.4479	3.5656	.5521	.9628	9.9836	20
50	.2728	9.4359	.2836	9.4527	3.5261	.5473	.9621	9.9832	10
16° 00′	.2756	9.4403	.2867	9.4575	3.4874	.5425	.9613	9.9828	**74° 00′**
10	.2784	9.4447	.2899	9.4622	3.4495	.5378	.9605	9.9825	50
20	.2812	9.4491	.2931	9.4669	3.4124	.5331	.9596	9.9821	40
30	.2840	9.4533	.2962	9.4716	3.3759	.5284	.9588	9.9817	30
40	.2868	9.4576	.2994	9.4762	3.3402	.5238	.9580	9.9814	20
50	.2896	9.4618	.3026	9.4808	3.3052	.5192	.9572	9.9810	10
17° 00′	.2924	9.4659	.3057	9.4853	3.2709	.5147	.9563	9.9806	**73° 00′**
10	.2952	9.4700	.3089	9.4898	3.2371	.5102	.9555	9.9802	50
20	.2979	9.4741	.3121	9.4943	3.2041	.5057	.9546	9.9798	40
30	.3007	9.4781	.3153	9.4987	3.1716	.5013	.9537	9.9794	30
40	.3035	9.4821	.3185	9.5031	3.1397	.4969	.9528	9.9790	20
50	.3062	9.4861	.3217	9.5075	3.1084	.4925	.9520	9.9786	10
18° 00′	.3090	9.4900	.3249	9.5118	3.0777	.4882	.9511	9.9782	**72° 00′**
	Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
18° 00′	.3090	9.4900	.3249	9.5118	3.0777	.4882	.9511	9.9782	72° 00′
10	.3118	9.4939	.3281	9.5161	3.0475	.4839	.9502	9.9778	50
20	.3145	9.4977	.3314	9.5203	3.0178	.4797	.9492	9.9774	40
30	.3173	9.5015	.3346	9.5245	2.9887	.4755	.9483	9.9770	30
40	.3201	9.5052	.3378	9.5287	2.9600	.4713	.9474	9.9765	20
50	.3228	9.5090	.3411	9.5329	2.9319	.4671	.9465	9.9761	10
19° 00′	.3256	9.5126	.3443	9.5370	2.9042	.4630	.9455	9.9757	71° 00′
10	.3283	9.5163	.3476	9.5411	2.8770	.4589	.9446	9.9752	50
20	.3311	9.5199	.3508	9.5451	2.8502	.4549	.9436	9.9748	40
30	.3338	9.5235	.3541	9.5491	2.8239	.4509	.9426	9.9743	30
40	.3365	9.5270	.3574	9.5531	2.7980	.4469	.9417	9.9739	20
50	.3393	9.5306	.3607	9.5571	2.7725	.4429	.9407	9.9734	10
20° 00′	.3420	9.5341	.3640	9.5611	2.7475	.4389	.9397	9.9730	70° 00′
10	.3448	9.5375	.3673	9.5650	2.7228	.4350	.9387	9.9725	50
20	.3475	9.5409	.3706	9.5689	2.6985	.4311	.9377	9.9721	40
30	.3502	9.5443	.3739	9.5727	2.6746	.4273	.9367	9.9716	30
40	.3529	9.5477	.3772	9.5766	2.6511	.4234	.9356	9.9711	20
50	.3557	9.5510	.3805	9.5804	2.6279	.4196	.9346	9.9706	10
21° 00′	.3584	9.5543	.3839	9.5842	2.6051	.4158	.9336	9.9702	69° 00′
10	.3611	9.5576	.3872	9.5879	2.5826	.4121	.9325	9.9697	50
20	.3638	9.5609	.3906	9.5917	2.5605	.4083	.9315	9.9692	40
30	.3665	9.5641	.3939	9.5954	2.5386	.4046	.9304	9.9687	30
40	.3692	9.5673	.3973	9.5991	2.5172	.4009	.9293	9.9682	20
50	.3719	9.5704	.4006	9.6028	2.4960	.3972	.9283	9.9677	10
22° 00′	.3746	9.5736	.4040	9.6064	2.4751	.3936	.9272	9.9672	68° 00′
10	.3773	9.5767	.4074	9.6100	2.4545	.3900	.9261	9.9667	50
20	.3800	9.5798	.4108	9.6136	2.4342	.3864	.9250	9.9661	40
30	.3827	9.5828	.4142	9.6172	2.4142	.3828	.9239	9.9656	30
40	.3854	9.5859	.4176	9.6208	2.3945	.3792	.9228	9.9651	20
50	.3881	9.5889	.4210	9.6243	2.3750	.3757	.9216	9.9646	10
23° 00′	.3907	9.5919	.4245	9.6279	2.3559	.3721	.9205	9.9640	67° 00′
10	.3934	9.5948	.4279	9.6314	2.3369	.3686	.9194	9.9635	50
20	.3961	9.5978	.4314	9.6348	2.3183	.3652	.9182	9.9629	40
30	.3987	9.6007	.4348	9.6383	2.2998	.3617	.9171	9.9624	30
40	.4014	9.6036	.4383	9.6417	2.2817	.3583	.9159	9.9618	20
50	.4041	9.6065	.4417	9.6452	2.2637	.3548	.9147	9.9613	10
24° 00′	.4067	9.6093	.4452	9.6486	2.2460	.3514	.9135	9.9607	66° 00′
10	.4094	9.6121	.4487	9.6520	2.2286	.3480	.9124	9.9602	50
20	.4120	9.6149	.4522	9.6553	2.2113	.3447	.9112	9.9596	40
30	.4147	9.6177	.4557	9.6587	2.1943	.3413	.9100	9.9590	30
40	.4173	9.6205	.4592	9.6620	2.1775	.3380	.9088	9.9584	20
50	.4200	9.6232	.4628	9.6654	2.1609	.3346	.9075	9.9579	10
25° 00′	.4226	9.6259	.4663	9.6687	2.1445	.3313	.9063	9.9573	65° 00′
10	.4253	9.6286	.4699	9.6720	2.1283	.3280	.9051	9.9567	50
20	.4279	9.6313	.4734	9.6752	2.1123	.3248	.9038	9.9561	40
30	.4305	9.6340	.4770	9.6785	2.0965	.3215	.9026	9.9555	30
40	.4331	9.6366	.4806	9.6817	2.0809	.3183	.9013	9.9549	20
50	.4358	9.6392	.4841	9.6850	2.0655	.3150	.9001	9.9543	10
26° 00′	.4384	9.6418	.4877	9.6882	2.0503	.3118	.8988	9.9537	64° 00′
10	.4410	9.6444	.4913	9.6914	2.0353	.3086	.8975	9.9530	50
20	.4436	9.6470	.4950	9.6946	2.0204	.3054	.8962	9.9524	40
30	.4462	9.6495	.4986	9.6977	2.0057	.3023	.8949	9.9518	30
40	.4488	9.6521	.5022	9.7009	1.9912	.2991	.8936	9.9512	20
50	.4514	9.6546	.5059	9.7040	1.9768	.2960	.8923	9.9505	10
27° 00′	.4540	9.6570	.5095	9.7072	1.9626	.2928	.8910	9.9499	63° 00′
	Value	Log	Value	Log	Value	Log	Value	Log	Degrees
	Cosine		Cotangent		Tangent		Sine		

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
27° 00′	.4540	9.6570	.5095	9.7072	1.9626	.2928	.8910	9.9499	63° 00′
10	.4566	9.6595	.5132	9.7103	1.9486	.2897	.8897	9.9492	50
20	.4592	9.6620	.5169	9.7134	1.9347	.2866	.8884	9.9486	40
30	.4617	9.6644	.5206	9.7165	1.9210	.2835	.8870	9.9479	30
40	.4643	9.6668	.5243	9.7196	1.9074	.2804	.8857	9.9473	20
50	.4669	9.6692	.5280	9.7226	1.8940	.2774	.8843	9.9466	10
28° 00′	.4695	9.6716	.5317	9.7257	1.8807	.2743	.8829	9.9459	62° 00′
10	.4720	9.6740	.5354	9.7287	1.8676	.2713	.8816	9.9453	50
20	.4746	9.6763	.5392	9.7317	1.8546	.2683	.8802	9.9446	40
30	.4772	9.6787	.5430	9.7348	1.8418	.2652	.8788	9.9439	30
40	.4797	9.6810	.5467	9.7378	1.8291	.2622	.8774	9.9432	20
50	.4823	9.6833	.5505	9.7408	1.8165	.2592	.8760	9.9425	10
29° 00′	.4848	9.6856	.5543	9.7438	1.8040	.2562	.8746	9.9418	61° 00′
10	.4874	9.6878	.5581	9.7467	1.7917	.2533	.8732	9.9411	50
20	.4899	9.6901	.5619	9.7497	1.7796	.2503	.8718	9.9404	40
30	.4924	9.6923	.5658	9.7526	1.7675	.2474	.8704	9.9397	30
40	.4950	9.6946	.5696	9.7556	1.7556	.2444	.8689	9.9390	20
50	.4975	9.6968	.5735	9.7585	1.7437	.2415	.8675	9.9383	10
30° 00′	.5000	9.6990	.5774	9.7614	1.7321	.2386	.8660	9.9375	60° 00′
10	.5025	9.7012	.5812	9.7644	1.7205	.2356	.8646	9.9368	50
20	.5050	9.7033	.5851	9.7673	1.7090	.2327	.8631	9.9361	40
30	.5075	9.7055	.5890	9.7701	1.6977	.2299	.8616	9.9353	30
40	.5100	9.7076	.5930	9.7730	1.6864	.2270	.8601	9.9346	20
50	.5125	9.7097	.5969	9.7759	1.6753	.2241	.8587	9.9338	10
31° 00′	.5150	9.7118	.6009	9.7788	1.6643	.2212	.8572	9.9331	59° 00′
10	.5175	9.7139	.6048	9.7816	1.6534	.2184	.8557	9.9323	50
20	.5200	9.7160	.6088	9.7845	1.6426	.2155	.8542	9.9315	40
30	.5225	9.7181	.6128	9.7873	1.6319	.2127	.8526	9.9308	30
40	.5250	9.7201	.6168	9.7902	1.6212	.2098	.8511	9.9300	20
50	.5275	9.7222	.6208	9.7930	1.6107	.2070	.8496	9.9292	10
32° 00′	.5299	9.7242	.6249	9.7958	1.6003	.2042	.8480	9.9284	58° 00′
10	.5324	9.7262	.6289	9.7986	1.5900	.2014	.8465	9.9276	50
20	.5348	9.7282	.6330	9.8014	1.5798	.1986	.8450	9.9268	40
30	.5373	9.7302	.6371	9.8042	1.5697	.1958	.8434	9.9260	30
40	.5398	9.7322	.6412	9.8070	1.5597	.1930	.8418	9.9252	20
50	.5422	9.7342	.6453	9.8097	1.5497	.1903	.8403	9.9244	10
33° 00′	.5446	9.7361	.6494	9.8125	1.5399	.1875	.8387	9.9236	57° 00′
10	.5471	9.7380	.6536	9.8153	1.5301	.1847	.8371	9.9228	50
20	.5495	9.7400	.6577	9.8180	1.5204	.1820	.8355	9.9219	40
30	.5519	9.7419	.6619	9.8208	1.5108	.1792	.8339	9.9211	30
40	.5544	9.7438	.6661	9.8235	1.5013	.1765	.8323	9.9203	20
50	.5568	9.7457	.6703	9.8263	1.4919	.1737	.8307	9.9194	10
34° 00′	.5592	9.7476	.6745	9.8290	1.4826	.1710	.8290	9.9186	56° 00′
10	.5616	9.7494	.6787	9.8317	1.4733	.1683	.8274	9.9177	50
20	.5640	9.7513	.6830	9.8344	1.4641	.1656	.8258	9.9169	40
30	.5664	9.7531	.6873	9.8371	1.4550	.1629	.8241	9.9160	30
40	.5688	9.7550	.6916	9.8398	1.4460	.1602	.8225	9.9151	20
50	.5712	9.7568	.6959	9.8425	1.4370	.1575	.8208	9.9142	10
35° 00′	.5736	9.7586	.7002	9.8452	1.4281	.1548	.8192	9.9134	55° 00′
10	.5760	9.7604	.7046	9.8479	1.4193	.1521	.8175	9.9125	50
20	.5783	9.7622	.7089	9.8506	1.4106	.1494	.8158	9.9116	40
30	.5807	9.7640	.7133	9.8533	1.4019	.1467	.8141	9.9107	30
40	.5831	9.7657	.7177	9.8559	1.3934	.1441	.8124	9.9098	20
50	.5854	9.7675	.7221	9.8586	1.3848	.1414	.8107	9.9089	10
36° 00′	.5878	9.7692	.7265	9.8613	1.3764	.1387	.8090	9.9080	54° 00′
	Value	Log	Value	Log	Value	Log	Value	Log	Degrees
	Cosine		Cotangent		Tangent		Sine		

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
36° 00′	.5878	9.7692	.7265	9.8613	1.3764	.1387	.8090	9.9080	54° 00′
10	.5901	9.7710	.7310	9.8639	1.3680	.1361	.8073	9.9070	50
20	.5925	9.7727	.7355	9.8666	1.3597	.1334	.8056	9.9061	40
30	.5948	9.7744	.7400	9.8692	1.3514	.1308	.8039	9.9052	30
40	.5972	9.7761	.7445	9.8718	1.3432	.1282	.8021	9.9042	20
50	.5995	9.7778	.7490	9.8745	1.3351	.1255	.8004	9.9033	10
37° 00′	.6018	9.7795	.7536	9.8771	1.3270	.1229	.7986	9.9023	53° 00′
10	.6041	9.7811	.7581	9.8797	1.3190	.1203	.7969	9.9014	50
20	.6065	9.7828	.7627	9.8824	1.3111	.1176	.7951	9.9004	40
30	.6088	9.7844	.7673	9.8850	1.3032	.1150	.7934	9.8995	30
40	.6111	9.7861	.7720	9.8876	1.2954	.1124	.7916	9.8985	20
50	.6134	9.7877	.7766	9.8902	1.2876	.1098	.7898	9.8975	10
38° 00′	.6157	9.7893	.7813	9.8928	1.2799	.1072	.7880	9.8965	52° 00′
10	.6180	9.7910	.7860	9.8954	1.2723	.1046	.7862	9.8955	50
20	.6202	9.7926	.7907	9.8980	1.2647	.1020	.7844	9.8945	40
30	.6225	9.7941	.7954	9.9006	1.2572	.0994	.7826	9.8935	30
40	.6248	9.7957	.8002	9.9032	1.2497	.0968	.7808	9.8925	20
50	.6271	9.7973	.8050	9.9058	1.2423	.0942	.7790	9.8915	10
39° 00′	.6293	9.7989	.8098	9.9084	1.2349	.0916	.7771	9.8905	51° 00′
10	.6316	9.8004	.8146	9.9110	1.2276	.0890	.7753	9.8895	50
20	.6338	9.8020	.8195	9.9135	1.2203	.0865	.7735	9.8884	40
30	.6361	9.8035	.8243	9.9161	1.2131	.0839	.7716	9.8874	30
40	.6383	9.8050	.8292	9.9187	1.2059	.0813	.7698	9.8864	20
50	.6406	9.8066	.8342	9.9212	1.1988	.0788	.7679	9.8853	10
40° 00′	.6428	9.8081	.8391	9.9238	1.1918	.0762	.7660	9.8843	50° 00′
10	.6450	9.8096	.8441	9.9264	1.1847	.0736	.7642	9.8832	50
20	.6472	9.8111	.8491	9.9289	1.1778	.0711	.7623	9.8821	40
30	.6494	9.8125	.8541	9.9315	1.1708	.0685	.7604	9.8810	30
40	.6517	9.8140	.8591	9.9341	1.1640	.0659	.7585	9.8800	20
50	.6539	9.8155	.8642	9.9366	1.1571	.0634	.7566	9.8789	10
41° 00′	.6561	9.8169	.8693	9.9392	1.1504	.0608	.7547	9.8778	49° 00′
10	.6583	9.8184	.8744	9.9417	1.1436	.0583	.7528	9.8767	50
20	.6604	9.8198	.8796	9.9443	1.1369	.0557	.7509	9.8756	40
30	.6626	9.8213	.8847	9.9468	1.1303	.0532	.7490	9.8745	30
40	.6648	9.8227	.8899	9.9494	1.1237	.0506	.7470	9.8733	20
50	.6670	9.8241	.8952	9.9519	1.1171	.0481	.7451	9.8722	10
42° 00′	.6691	9.8255	.9004	9.9544	1.1106	.0456	.7431	9.8711	48° 00′
10	.6713	9.8269	.9057	9.9570	1.1041	.0430	.7412	9.8699	50
20	.6734	9.8283	.9110	9.9595	1.0977	.0405	.7392	9.8688	40
30	.6756	9.8297	.9163	9.9621	1.0913	.0379	.7373	9.8676	30
40	.6777	9.8311	.9217	9.9646	1.0850	.0354	.7353	9.8665	20
50	.6799	9.8324	.9271	9.9671	1.0786	.0329	.7333	9.8653	10
43° 00′	.6820	9.8338	.9325	9.9697	1.0724	.0303	.7314	9.8641	47° 00′
10	.6841	9.8351	.9380	9.9722	1.0661	.0278	.7294	9.8629	50
20	.6862	9.8365	.9435	9.9747	1.0599	.0253	.7274	9.8618	40
30	.6884	9.8378	.9490	9.9772	1.0538	.0228	.7254	9.8606	30
40	.6905	9.8391	.9545	9.9798	1.0477	.0202	.7234	9.8594	20
50	.6926	9.8405	.9601	9.9823	1.0416	.0177	.7214	9.8582	10
44° 00′	.6947	9.8418	.9657	9.9848	1.0355	.0152	.7193	9.8569	46° 00′
10	.6967	9.8431	.9713	9.9874	1.0295	.0126	.7173	9.8557	50
20	.6988	9.8444	.9770	9.9899	1.0235	.0101	.7153	9.8545	40
30	.7009	9.8457	.9827	9.9924	1.0176	.0076	.7133	9.8532	30
40	.7030	9.8469	.9884	9.9949	1.0117	.0051	.7112	9.8520	20
50	.7050	9.8482	.9942	9.9975	1.0058	.0025	.7092	9.8507	10
45° 00′	.7071	9.8495	1.0000	.0000	1.0000	.0000	.7071	9.8495	45° 00′
	Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees

Table III

CONVERSION TABLE

Radian—Degrees, Degrees—Radians

RADIANS TO DEGREES

Radians	Degrees
1	57.2958
2	114.5916
3	171.8873
4	229.1831
5	286.4789
6	343.7747
7	401.0705
8	458.3662
9	515.6620
10	572.9578

Note: To obtain $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, etc., of any radian value divide the corresponding degree value by 10, 100, 1000, etc.

To obtain 10 times, 100 times, 1000 times, etc., of any radian value multiply the corresponding degree value by 10, 100, 1000, etc.

Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log	
36° 00′	.5878	9.7692	.7265	9.8613	1.3764	.1387	.8090	9.9080	54° 00′
10	.5901	9.7710	.7310	9.8639	1.3680	.1361	.8073	9.9070	50
20	.5925	9.7727	.7355	9.8666	1.3597	.1334	.8056	9.9061	40
30	.5948	9.7744	.7400	9.8692	1.3514	.1308	.8039	9.9052	30
40	.5972	9.7761	.7445	9.8718	1.3432	.1282	.8021	9.9042	20
50	.5995	9.7778	.7490	9.8745	1.3351	.1255	.8004	9.9033	10
37° 00′	.6018	9.7795	.7536	9.8771	1.3270	.1229	.7986	9.9023	53° 00′
10	.6041	9.7811	.7581	9.8797	1.3190	.1203	.7969	9.9014	50
20	.6065	9.7828	.7627	9.8824	1.3111	.1176	.7951	9.9004	40
30	.6088	9.7844	.7673	9.8850	1.3032	.1150	.7934	9.8995	30
40	.6111	9.7861	.7720	9.8876	1.2954	.1124	.7916	9.8985	20
50	.6134	9.7877	.7766	9.8902	1.2876	.1098	.7898	9.8975	10
38° 00′	.6157	9.7893	.7813	9.8928	1.2799	.1072	.7880	9.8965	52° 00′
10	.6180	9.7910	.7860	9.8954	1.2723	.1046	.7862	9.8955	50
20	.6202	9.7926	.7907	9.8980	1.2647	.1020	.7844	9.8945	40
30	.6225	9.7941	.7954	9.9006	1.2572	.0994	.7826	9.8935	30
40	.6248	9.7957	.8002	9.9032	1.2497	.0968	.7808	9.8925	20
50	.6271	9.7973	.8050	9.9058	1.2423	.0942	.7790	9.8915	10
39° 00′	.6293	9.7989	.8098	9.9084	1.2349	.0916	.7771	9.8905	51° 00′
10	.6316	9.8004	.8146	9.9110	1.2276	.0890	.7753	9.8895	50
20	.6338	9.8020	.8195	9.9135	1.2203	.0865	.7735	9.8884	40
30	.6361	9.8035	.8243	9.9161	1.2131	.0839	.7716	9.8874	30
40	.6383	9.8050	.8292	9.9187	1.2059	.0813	.7698	9.8864	20
50	.6406	9.8066	.8342	9.9212	1.1988	.0788	.7679	9.8853	10
40° 00′	.6428	9.8081	.8391	9.9238	1.1918	.0762	.7660	9.8843	50° 00′
10	.6450	9.8096	.8441	9.9264	1.1847	.0736	.7642	9.8832	50
20	.6472	9.8111	.8491	9.9289	1.1778	.0711	.7623	9.8821	40
30	.6494	9.8125	.8541	9.9315	1.1708	.0685	.7604	9.8810	30
40	.6517	9.8140	.8591	9.9341	1.1640	.0659	.7585	9.8800	20
50	.6539	9.8155	.8642	9.9366	1.1571	.0634	.7566	9.8789	10
41° 00′	.6561	9.8169	.8693	9.9392	1.1504	.0608	.7547	9.8778	49° 00′
10	.6583	9.8184	.8744	9.9417	1.1436	.0583	.7528	9.8767	50
20	.6604	9.8198	.8796	9.9443	1.1369	.0557	.7509	9.8756	40
30	.6626	9.8213	.8847	9.9468	1.1303	.0532	.7490	9.8745	30
40	.6648	9.8227	.8899	9.9494	1.1237	.0506	.7470	9.8733	20
50	.6670	9.8241	.8952	9.9519	1.1171	.0481	.7451	9.8722	10
42° 00′	.6691	9.8255	.9004	9.9544	1.1106	.0456	.7431	9.8711	48° 00′
10	.6713	9.8269	.9057	9.9570	1.1041	.0430	.7412	9.8699	50
20	.6734	9.8283	.9110	9.9595	1.0977	.0405	.7392	9.8688	40
30	.6756	9.8297	.9163	9.9621	1.0913	.0379	.7373	9.8676	30
40	.6777	9.8311	.9217	9.9646	1.0850	.0354	.7353	9.8665	20
50	.6799	9.8324	.9271	9.9671	1.0786	.0329	.7333	9.8653	10
43° 00′	.6820	9.8338	.9325	9.9697	1.0724	.0303	.7314	9.8641	47° 00′
10	.6841	9.8351	.9380	9.9722	1.0661	.0278	.7294	9.8629	50
20	.6862	9.8365	.9435	9.9747	1.0599	.0253	.7274	9.8618	40
30	.6884	9.8378	.9490	9.9772	1.0538	.0228	.7254	9.8606	30
40	.6905	9.8391	.9545	9.9798	1.0477	.0202	.7234	9.8594	20
50	.6926	9.8405	.9601	9.9823	1.0416	.0177	.7214	9.8582	10
44° 00′	.6947	9.8418	.9657	9.9848	1.0355	.0152	.7193	9.8569	46° 00′
10	.6967	9.8431	.9713	9.9874	1.0295	.0126	.7173	9.8557	50
20	.6988	9.8444	.9770	9.9899	1.0235	.0101	.7153	9.8545	40
30	.7009	9.8457	.9827	9.9924	1.0176	.0076	.7133	9.8532	30
40	.7030	9.8469	.9884	9.9949	1.0117	.0051	.7112	9.8520	20
50	.7050	9.8482	.9942	9.9975	1.0058	.0025	.7092	9.8507	10
45° 00′	.7071	9.8495	1.0000	.0000	1.0000	.0000	.7071	9.8495	45° 00′
	Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees

Table IV

NATURAL, OR NAPERIAN, LOGARITHMS *

TO FIND THE NATURAL LOGARITHM OF A NUMBER WHICH IS 1/10 OR 10 TIMES A NUMBER WHOSE LOGARITHM IS GIVEN, SUBTRACT FROM OR ADD TO THE GIVEN LOGARITHM THE LOGARITHM OF 10.

* From *Handbook of Chemistry and Physics.* Cleveland, Ohio: The Chemical Rubber Publishing Company.

Degrees						Min.		Sec.	
0°	0.00000	60°	1.04720	120°	2.09440	0′	0.00000	0″	0.00000
1	0.01745	61	1.06465	121	2.11185	1	0.00029	1	0.00000
2	0.03491	62	1.08210	122	2.12930	2	0.00058	2	0.00001
3	0.05236	63	1.09956	123	2.14676	3	0.00087	3	0.00001
4	0.06981	64	1.11701	124	2.16421	4	0.00116	4	0.00002
5	0.08727	65	1.13446	125	2.18166	5	0.00145	5	0.00002
6	0.10472	66	1.15192	126	2.19911	6	0.00175	6	0.00003
7	0.12217	67	1.16937	127	2.21657	7	0.00204	7	0.00003
8	0.13963	68	1.18682	128	2.23402	8	0.00233	8	0.00004
9	0.15708	69	1.20428	129	2.25147	9	0.00262	9	0.00004
10	0.17453	70	1.22173	130	2.26893	10	0.00291	10	0.00005
11	0.19199	71	1.23918	131	2.28638	11	0.00320	11	0.00005
12	0.20944	72	1.25664	132	2.30383	12	0.00349	12	0.00006
13	0.22689	73	1.27409	133	2.32129	13	0.00378	13	0.00006
14	0.24435	74	1.29154	134	2.33874	14	0.00407	14	0.00007
15	0.26180	75	1.30900	135	2.35619	15	0.00436	15	0.00007
16	0.27925	76	1.32645	136	2.37365	16	0.00465	16	0.00008
17	0.29671	77	1.34390	137	2.39110	17	0.00495	17	0.00008
18	0.31416	78	1.36136	138	2.40855	18	0.00524	18	0.00009
19	0.33161	79	1.37881	139	2.42601	19	0.00553	19	0.00009
20	0.34907	80	1.39626	140	2.44346	20	0.00582	20	0.00010
21	0.36652	81	1.41372	141	2.46091	21	0.00611	21	0.00010
22	0.38397	82	1.43117	142	2.47837	22	0.00640	22	0.00011
23	0.40143	83	1.44862	143	2.49582	23	0.00669	23	0.00011
24	0.41888	84	1.46608	144	2.51327	24	0.00698	24	0.00012
25	0.43633	85	1.48353	145	2.53073	25	0.00727	25	0.00012
26	0.45379	86	1.50098	146	2.54818	26	0.00756	26	0.00013
27	0.47124	87	1.51844	147	2.56563	27	0.00785	27	0.00013
28	0.48869	88	1.53589	148	2.58309	28	0.00814	28	0.00014
29	0.50615	89	1.55334	149	2.60054	29	0.00844	29	0.00014
30	0.52360	90	1.57080	150	2.61799	30	0.00873	30	0.00015
31	0.54105	91	1.58825	151	2.63545	31	0.00902	31	0.00015
32	0.55851	92	1.60570	152	2.65290	32	0.00931	32	0.00016
33	0.57596	93	1.62316	153	2.67035	33	0.00960	33	0.00016
34	0.59341	94	1.64061	154	2.68781	34	0.00989	34	0.00016
35	0.61087	95	1.65806	155	2.70526	35	0.01018	35	0.00017
36	0.62832	96	1.67552	156	2.72271	36	0.01047	36	0.00017
37	0.64577	97	1.69297	157	2.74017	37	0.01076	37	0.00018
38	0.66323	98	1.71042	158	2.75762	38	0.01105	38	0.00018
39	0.68068	99	1.72788	159	2.77507	39	0.01134	39	0.00019
40	0.69813	100	1.74533	160	2.79253	40	0.01164	40	0.00019
41	0.71559	101	1.76278	161	2.80998	41	0.01193	41	0.00020
42	0.73304	102	1.78024	162	2.82743	42	0.01222	42	0.00020
43	0.75049	103	1.79769	163	2.84489	43	0.01251	43	0.00021
44	0.76794	104	1.81514	164	2.86234	44	0.01280	44	0.00021
45	0.78540	105	1.83260	165	2.87979	45	0.01309	45	0.00022
46	0.80285	106	1.85005	166	2.89725	46	0.01338	46	0.00022
47	0.82030	107	1.86750	167	2.91470	47	0.01367	47	0.00023
48	0.83776	108	1.88496	168	2.93215	48	0.01396	48	0.00023
49	0.85521	109	1.90241	169	2.94961	49	0.01425	49	0.00024
50	0.87266	110	1.91986	170	2.96706	50	0.01454	50	0.00024
51	0.89012	111	1.93732	171	2.98451	51	0.01484	51	0.00025
52	0.90757	112	1.95477	172	3.00197	52	0.01513	52	0.00025
53	0.92502	113	1.97222	173	3.01942	53	0.01542	53	0.00026
54	0.94248	114	1.98968	174	3.03687	54	0.01571	54	0.00026
55	0.95993	115	2.00713	175	3.05433	55	0.01600	55	0.00027
56	0.97738	116	2.02458	176	3.07178	56	0.01629	56	0.00027
57	0.99484	117	2.04204	177	3.08923	57	0.01658	57	0.00028
58	1.01229	118	2.05949	178	3.10669	58	0.01687	58	0.00028
59	1.02974	119	2.07694	179	3.12414	59	0.01716	59	0.00029
60	1.04720	120	2.09440	180	3.14159	60	0.01745	60	0.00029
Degrees						Min.		Sec.	

B 1.00–10.09 (Concluded)

N	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
5.5	1.7 0475	0656	0838	1019	1199	1380	1560	1740	1919	2098
5.6	2277	2455	2633	2811	2988	3166	3342	3519	3695	3871
5.7	4047	4222	4397	4572	4746	4920	5094	5267	5440	5613
5.8	5786	5958	6130	6302	6473	6644	6815	6985	7156	7326
5.9	7495	7665	7834	8002	8171	8339	8507	8675	8842	9009
6.0	9176	9342	9509	9675	9840	*0006	*0171	*0336	*0500	*0665
6.1	1.8 0829	0993	1156	1319	1482	1645	1808	1970	2132	2294
6.2	2455	2616	2777	2938	3098	3258	3418	3578	3737	3896
6.3	4055	4214	4372	4530	4688	4845	5003	5160	5317	5473
6.4	5630	5786	5942	6097	6253	6408	6563	6718	6872	7026
6.5	7180	7334	7487	7641	7794	7947	8099	8251	8403	8555
6.6	8707	8858	9010	9160	9311	9462	9612	9762	9912	*0061
6.7	1.9 0211	0360	0509	0658	0806	0954	1102	1250	1398	1545
6.8	1692	1839	1986	2132	2279	2425	2571	2716	2862	3007
6.9	3152	3297	3442	3586	3730	3874	4018	4162	4305	4448
7.0	4591	4734	4876	5019	5161	5303	5445	5586	5727	5869
7.1	6009	6150	6291	6431	6571	6711	6851	6991	7130	7269
7.2	7408	7547	7685	7824	7962	8100	8238	8376	8513	8650
7.3	8787	8924	9061	9198	9334	9470	9606	9742	9877	*0013
7.4	2.0 0148	0283	0418	0553	0687	0821	0956	1089	1223	1357
7.5	1490	1624	1757	1890	2022	2155	2287	2419	2551	2683
7.6	2815	2946	3078	3209	3340	3471	3601	3732	3862	3992
7.7	4122	4252	4381	4511	4640	4769	4898	5027	5156	5284
7.8	5412	5540	5668	5796	5924	6051	6179	6306	6433	6560
7.9	6686	6813	6939	7065	7191	7317	7443	7568	7694	7819
8.0	7944	8069	8194	8318	8443	8567	8691	8815	8939	9063
8.1	9186	9310	9433	9556	9679	9802	9924	*0047	*0169	*0291
8.2	2.1 0413	0535	0657	0779	0900	1021	1142	1263	1384	1505
8.3	1626	1746	1866	1986	2106	2226	2346	2465	2585	2704
8.4	2823	2942	3061	3180	3298	3417	3535	3653	3771	3889
8.5	4007	4124	4242	4359	4476	4593	4710	4827	4943	5060
8.6	5176	5292	5409	5524	5640	5756	5871	5987	6102	6217
8.7	6332	6447	6562	6677	6791	6905	7020	7134	7248	7361
8.8	7475	7589	7702	7816	7929	8042	8155	8267	8380	8493
8.9	8605	8717	8830	8942	9054	9165	9277	9389	9500	9611
9.0	9722	9834	9944	*0055	*0166	*0276	*0387	*0497	*0607	*0717
9.1	2.2 0827	0937	1047	1157	1266	1375	1485	1594	1703	1812
9.2	1920	2029	2138	2246	2354	2462	2570	2678	2786	2894
9.3	3001	3109	3216	3324	3431	3538	3645	3751	3858	3965
9.4	4071	4177	4284	4390	4496	4601	4707	4813	4918	5024
9.5	5129	5234	5339	5444	5549	5654	5759	5863	5968	6072
9.6	6176	6280	6384	6488	6592	6696	6799	6903	7006	7109
9.7	7213	7316	7419	7521	7624	7727	7829	7932	8034	8136
9.8	8238	8340	8442	8544	8646	8747	8849	8950	9051	9152
9.9	9253	9354	9455	9556	9657	9757	9858	9958	*0058	*0158
10.0	2.3 0259	0358	0458	0558	0658	0757	0857	0956	1055	1154

C 10–99

N	0	1	2	3	4	5	6	7	8	9
1	2.30259	39790	48491	56495	63906	70805	77259	83321	89037	94444
2	99573	*04452	*09104	*13549	*17805	*21888	*25810	*29584	*33220	*36730
3	3.40120	43399	46574	49651	52636	55535	58352	61092	63759	66356
4	68888	71357	73767	76120	78419	80666	82864	85015	87120	89182
5	91202	93183	95124	97029	98898	*00733	*02535	*04305	*06044	*07754
6	4.09434	11087	12713	14313	15888	17439	18965	20469	21951	23411
7	24850	26268	27667	29046	30407	31749	33073	34381	35671	36945
8	38203	39445	40672	41884	43082	44265	45435	46591	47734	48864
9	49981	51086	52179	53260	54329	55388	56435	57471	58497	59512

D 100-1109

N	0	1	2	3	4	5	6	7	8	9
10	4.6 0517	1512	2497	3473	4439	5396	6344	7283	8213	9135
11	4.7 0048	0953	1850	2739	3620	4493	5359	6217	7068	7912
12	8749	9579	*0402	*1218	*2028	*2831	*3628	*4419	*5203	*5981
13	4.8 6753	7520	8280	9035	9784	*0527	*1265	*1998	*2725	*3447
14	4.9 4164	4876	5583	6284	6981	7673	8361	9043	9721	*0395
15	5.0 1064	1728	2388	3044	3695	4343	4986	5625	6260	6890
16	7517	8140	8760	9375	9987	*0595	*1199	*1799	*2396	*2990
17	5.1 3580	4166	4749	5329	5906	6479	7048	7615	8178	8739
18	9296	9850	*0401	*0949	*1494	*2036	*2575	*3111	*3644	*4175
19	5.2 4702	5227	5750	6269	6786	7300	7811	8320	8827	9330
20	9832	*0330	*0827	*1321	*1812	*2301	*2788	*3272	*3754	*4233
21	5.3 4711	5186	5659	6129	6598	7064	7528	7990	8450	8907
22	9363	9816	*0268	*0717	*1165	*1610	*2053	*2495	*2935	*3372
23	5.4 3808	4242	4674	5104	5532	5959	6383	6806	7227	7646
24	8064	8480	8894	9306	9717	*0126	*0533	*0939	*1343	*1745
25	5.5 2146	2545	2943	3339	3733	4126	4518	4908	5296	5683
26	6068	6452	6834	7215	7595	7973	8350	8725	9099	9471
27	9842	*0212	*0580	*0947	*1313	*1677	*2040	*2402	*2762	*3121
28	5.6 3479	3835	4191	4545	4897	5249	5599	5948	6296	6643
29	6988	7332	7675	8017	8358	8698	9036	9373	9709	*0044
30	5.7 0378	0711	1043	1373	1703	2031	2359	2685	3010	3334
31	3657	3979	4300	4620	4939	5257	5574	5890	6205	6519
32	6832	7144	7455	7765	8074	8383	8690	8996	9301	9606
33	9909	*0212	*0513	*0814	*1114	*1413	*1711	*2008	*2305	*2600
34	5.8 2895	3188	3481	3773	4064	4354	4644	4932	5220	5507
35	5793	6079	6363	6647	6930	7212	7493	7774	8053	8332
36	8610	8888	9164	9440	9715	9990	*0263	*0536	*0808	*1080
37	5.9 1350	1620	1889	2158	2426	2693	2959	3225	3489	3754
38	4017	4280	4542	4803	5064	5324	5584	5842	6101	6358
39	6615	6871	7126	7381	7635	7889	8141	8394	8645	8896
40	9146	9396	9645	9894	*0141	*0389	*0635	*0881	*1127	*1372
41	6.0 1616	1859	2102	2345	2587	2828	3069	3309	3548	3787
42	4025	4263	4501	4737	4973	5209	5444	5678	5912	6146
43	6379	6611	6843	7074	7304	7535	7764	7993	8222	8450
44	8677	8904	9131	9357	9582	9807	*0032	*0256	*0479	*0702
45	6.1 0925	1147	1368	1589	1810	2030	2249	2468	2687	2905
46	3123	3340	3556	3773	3988	4204	4419	4633	4847	5060
47	5273	5486	5698	5910	6121	6331	6542	6752	6961	7170
48	7379	7587	7794	8002	8208	8415	8621	8826	9032	9236
49	9441	9644	9848	*0051	*0254	*0456	*0658	*0859	*1060	*1261
50	6.2 1461	1661	1860	2059	2258	2456	2654	2851	3048	3245
51	3441	3637	3832	4028	4222	4417	4611	4804	4998	5190
52	5383	5575	5767	5958	6149	6340	6530	6720	6910	7099
53	7288	7476	7664	7852	8040	8227	8413	8600	8786	8972
54	9157	9342	9527	9711	9895	*0079	*0262	*0445	*0628	*0810
55	6.3 0992	1173	1355	1536	1716	1897	2077	2257	2436	2615
56	2794	2972	3150	3328	3505	3683	3859	4036	4212	4388
57	4564	4739	4914	5089	5263	5437	5611	5784	5957	6130
58	6303	6475	6647	6819	6990	7161	7332	7502	7673	7843
59	8012	8182	8351	8519	8688	8856	9024	9192	9359	9526
60	9693	9859	*0026	*0192	*0357	*0523	*0688	*0853	*1017	*1182
N	0	1	2	3	4	5	6	7	8	9

D 100-1109 (Concluded)

N	0	1	2	3	4	5	6	7	8	9
60	6.3 9693	9859	*0026	*0192	*0357	*0523	*0688	*0853	*1017	*1182
61	6.4 1346	1510	1673	1836	1999	2162	2325	2487	2649	2811
62	2972	3133	3294	3455	3615	3775	3935	4095	4254	4413
63	4572	4731	4889	5047	5205	5362	5520	5677	5834	5990
64	6147	6303	6459	6614	6770	6925	7080	7235	7389	7543
65	7697	7851	8004	8168	8311	8464	8616	8768	8920	9072
66	9224	9375	9527	9677	9828	9979	*0129	*0279	*0429	*0678
67	6.5 0728	0877	1026	1175	1323	1471	1619	1767	1915	2062
68	2209	2356	2503	2649	2796	2942	3088	3233	3379	3524
69	3669	3814	3959	4103	4247	4391	4535	4679	4822	4965
70	5108	5251	5393	5536	5678	5820	5962	6103	6244	6386
71	6526	6667	6808	6948	7088	7228	7368	7508	7647	7786
72	7925	8064	8203	8341	8479	8617	8755	8893	9030	9167
73	9304	9441	9578	9715	9851	9987	*0123	*0259	*0394	*0530
74	6.6 0665	0800	0935	1070	1204	1338	1473	1607	1740	1874
75	2007	2141	2274	2407	2539	2672	2804	2936	3068	3200
76	3332	3463	3595	3726	3857	3988	4118	4249	4379	4509
77	4639	4769	4898	5028	5157	5286	5415	5544	5673	5801
78	5929	6058	6185	6313	6441	6568	6696	6823	6950	7077
79	7203	7330	7456	7582	7708	7834	7960	8085	8211	8336
80	8461	8586	8711	8835	8960	9084	9208	9332	9456	9580
81	9703	9827	9950	*0073	*0196	*0319	*0441	*0564	*0686	*0808
82	6.7 0930	1052	1174	1296	1417	1538	1659	1780	1901	2022
83	2143	2263	2383	2503	2623	2743	2863	2982	3102	3221
84	3340	3459	3578	3697	3815	3934	4052	4170	4288	4406
85	4524	4641	4759	4876	4993	5110	5227	5344	5460	5577
86	5693	5809	5926	6041	6157	6273	6388	6504	6619	6734
87	6849	6964	7079	7194	7308	7422	7537	7651	7765	7878
88	7992	8106	8219	8333	8446	8559	8672	8784	8897	9010
89	9122	9234	9347	9459	9571	9682	9794	9906	*0017	*0128
90	6.8 0239	0351	0461	0572	0683	0793	0904	1014	1124	1235
91	1344	1454	1564	1674	1783	1892	2002	2111	2220	2329
92	2437	2546	2655	2763	2871	2979	3087	3195	3303	3411
93	3518	3626	3733	3841	3948	4055	4162	4268	4375	4482
94	4588	4694	4801	4907	5013	5118	5224	5330	5435	5541
95	5646	5751	5857	5961	6066	6171	6276	6380	6485	6589
96	6693	6797	6901	7005	7109	7213	7316	7420	7523	7626
97	7730	7833	7936	8038	8141	8244	8346	8449	8551	8653
98	8755	8857	8959	9061	9163	9264	9366	9467	9568	9669
99	9770	9871	9972	*0073	*0174	*0274	*0375	*0475	*0575	*0675
100	6.9 0776	0875	0975	1075	1175	1274	1374	1473	1572	1672
101	1771	1870	1968	2067	2166	2264	2363	2461	2560	2658
102	2756	2854	2952	3049	3147	3245	3342	3440	3537	3634
103	3731	3828	3925	4022	4119	4216	4312	4409	4505	4601
104	4698	4794	4890	4986	5081	5177	5273	6368	5464	6559
105	5655	5750	5845	5940	6035	6130	6224	6319	6414	6508
106	6602	6697	6791	6885	6979	7073	7167	7261	7354	7448
107	7541	7635	7728	7821	7915	8008	8101	8193	8286	8379
108	8472	8564	8657	8749	8841	8934	9026	9118	9210	9302
109	9393	9485	9577	9668	9760	9851	9942	*0033	*0125	*0216
110	7.0 0307	0397	0488	0579	0670	0760	0851	0941	1031	1121
N	0	1	2	3	4	5	6	7	8	9

(Moving the decimal point *one* place in N requires a corresponding move of *two* places in N^2.)

N	N^2: 0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0001	.0004	.0009	.0016	.0025	.0036	.0049	.0064	.0081
0.1	.0100	.0121	.0144	.0169	.0196	.0225	.0256	.0289	.0324	.0361
0.2	.0400	.0441	.0484	.0529	.0576	.0625	.0676	.0729	.0784	.0841
0.3	.0900	.0961	.1024	.1089	.1156	.1225	.1296	.1369	.1444	.1521
0.4	.1600	.1681	.1764	.1849	.1936	.2025	.2116	.2209	.2304	.2401
0.5	.2500	.2601	.2704	.2809	.2916	.3025	.3136	.3249	.3364	.3481
0.6	.3600	.3721	.3844	.3969	.4096	.4225	.4356	.4489	.4624	.4761
0.7	.4900	.5041	.5184	.5329	.5476	.5625	.5776	.5929	.6084	.6241
0.8	.6400	.6561	.6724	.6889	.7056	.7225	.7396	.7569	.7744	.7921
0.9	.8100	.8281	.8464	.8649	.8836	.9025	.9216	.9409	.9604	.9801
1.0	1.000	1.020	1.040	1.061	1.082	1.103	1.124	1.145	1.166	1.188
1.1	1.210	1.232	1.254	1.277	1.300	1.323	1.346	1.369	1.392	1.416
1.2	1.440	1.464	1.488	1.513	1.538	1.563	1.588	1.613	1.638	1.664
1.3	1.690	1.716	1.742	1.769	1.796	1.823	1.850	1.877	1.904	1.932
1.4	1.960	1.988	2.016	2.045	2.074	2.103	2.132	2.161	2.190	2.220
1.5	2.250	2.280	2.310	2.341	2.372	2.403	2.434	2.465	2.496	2.528
1.6	2.560	2.592	2.624	2.657	2.690	2.723	2.756	2.789	2.822	2.856
1.7	2.890	2.924	2.958	2.993	3.028	3.063	3.098	3.133	3.168	3.204
1.8	3.240	3.276	3.312	3.349	3.386	3.423	3.460	3.497	3.534	3.572
1.9	3.610	3.648	3.686	3.725	3.764	3.803	3.842	3.881	3.920	3.960
2.0	4.000	4.040	4.080	4.121	4.162	4.203	4.244	4.285	4.326	4.368
2.1	4.410	4.452	4.494	4.537	4.580	4.623	4.666	4.709	4.752	4.796
2.2	4.840	4.884	4.928	4.973	5.018	5.063	5.108	5.153	5.198	5.244
2.3	5.290	5.336	5.382	5.429	5.476	5.523	5.570	5.617	5.664	5.712
2.4	5.760	5.808	5.856	5.905	5.954	6.003	6.052	6.101	6.150	6.200
2.5	6.250	6.300	6.350	6.401	6.452	6.503	6.554	6.605	6.656	6.708
2.6	6.760	6.812	6.864	6.917	6.970	7.023	7.076	7.129	7.182	7.236
2.7	7.290	7.344	7.398	7.453	7.508	7.563	7.618	7.673	7.728	7.784
2.8	7.840	7.896	7.952	8.009	8.066	8.123	8.180	8.237	8.294	8.352
2.9	8.410	8.468	8.526	8.585	8.644	8.703	8.762	8.821	8.880	8.940
3.0	9.000	9.060	9.120	9.181	9.242	9.303	9.364	9.425	9.486	9.548
3.1	9.610	9.672	9.734	9.797	9.860	9.923	9.986	10.05	10.11	10.18
3.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36
3.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40
4.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91

(Moving the decimal point *one* place in N requires a corresponding move of *two* places in N^2.)

N	N^2 0	1	2	3	4	5	6	7	8	9
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91
5.1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94
5.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98
5.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05
5.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25
5.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38
5.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52
5.8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69
5.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09
6.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32
6.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56
6.3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83
6.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43
6.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76
6.7	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	45.97	46.10
6.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47
6.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27
7.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70
7.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14
7.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61
7.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61
7.6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14
7.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68
7.8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25
7.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45
8.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08
8.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72
8.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39
8.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79
8.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52
8.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26
8.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03
8.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63
9.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46
9.2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30
9.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17
9.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97
9.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90
9.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84
9.8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81
9.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80

Table VI

DECIMAL EQUIVALENTS OF COMMON FRACTIONS

$\frac{1}{64} = 0.015625$	$\frac{17}{64} = 0.265625$	$\frac{33}{64} = 0.515625$	$\frac{25}{32} = 0.78125$
$\frac{1}{32} = 0.03125$	$\frac{9}{32} = 0.28125$	$\frac{17}{32} = 0.53125$	$\frac{51}{64} = 0.796875$
$\frac{3}{64} = 0.046875$	$\frac{19}{64} = 0.296875$	$\frac{35}{64} = 0.546875$	$\frac{13}{16} = 0.8125$
$\frac{1}{16} = 0.0625$	$\frac{5}{16} = 0.3125$	$\frac{9}{16} = 0.5625$	$\frac{53}{64} = 0.828125$
$\frac{5}{64} = 0.078125$	$\frac{21}{64} = 0.328125$	$\frac{37}{64} = 0.578125$	$\frac{27}{32} = 0.84375$
$\frac{3}{32} = 0.09375$	$\frac{11}{32} = 0.34375$	$\frac{19}{32} = 0.59375$	$\frac{55}{64} = 0.859375$
$\frac{7}{64} = 0.109375$	$\frac{23}{64} = 0.359375$	$\frac{5}{8} = 0.625$	$\frac{7}{8} = 0.875$
$\frac{1}{8} = 0.1250$	$\frac{3}{8} = 0.375$	$\frac{41}{64} = 0.640625$	$\frac{57}{64} = 0.890625$
$\frac{9}{64} = 0.140625$	$\frac{25}{64} = 0.390625$	$\frac{21}{32} = 0.65625$	$\frac{29}{32} = 0.90625$
$\frac{5}{32} = 0.15625$	$\frac{13}{32} = 0.40625$	$\frac{43}{64} = 0.671875$	$\frac{59}{64} = 0.921875$
$\frac{11}{64} = 0.171875$	$\frac{27}{64} = 0.421875$	$\frac{11}{16} = 0.6875$	$\frac{15}{16} = 0.9375$
$\frac{3}{16} = 0.1875$	$\frac{7}{16} = 0.4375$	$\frac{45}{64} = 0.703125$	$\frac{61}{64} = 0.953125$
$\frac{13}{64} = 0.203125$	$\frac{29}{64} = 0.453125$	$\frac{23}{32} = 0.71875$	$\frac{31}{32} = 0.96875$
$\frac{7}{32} = 0.21875$	$\frac{15}{32} = 0.46875$	$\frac{47}{64} = 0.734375$	$\frac{63}{64} = 0.984375$
$\frac{15}{64} = 0.234375$	$\frac{31}{64} = 0.484375$	$\frac{3}{4} = 0.750$	$\frac{64}{64} = 1.0000$
$\frac{1}{4} = 0.250$	$\frac{1}{2} = 0.50$	$\frac{49}{64} = 0.765625$	

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